

THE MOTIVIC DONALDSON–THOMAS INVARIANTS OF (-2) -CURVES

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ABSTRACT. In this paper we calculate the motivic Donaldson–Thomas invariants for (-2) -curves arising from 3-fold flopping contractions in the minimal model programme. We translate this geometric situation into the machinery of [29], and using the results and framework of [7] we describe the monodromy on these invariants. In particular, in contrast to all existing known Donaldson–Thomas invariants for small resolutions of Gorenstein singularities these monodromy operations are nontrivial.

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1. INTRODUCTION

Motivic Donaldson–Thomas invariants were introduced in [29], as a generalisation of the classical theory of Donaldson–Thomas invariants initiated in [40]. At the same time Joyce ([15], [16], [17], [18], [19], [20], [21]) and Joyce and Song [22] rigorously extended the theory of classical Donaldson–Thomas theory to take care of the technicalities involved in dealing with strictly semistable coherent sheaves on Calabi–Yau 3-folds, and in this framework formulated a deep integrality conjecture regarding the resulting Donaldson–Thomas invariants. Assuming the more ambitious framework of [29], integrality properties of generalised Donaldson–Thomas invariants are conjecturally obtained by applying Euler characteristic to motivic Donaldson–Thomas invariants, after multiplication by the motive \mathbb{C}^* ; such statements are supposed to be a shadow of the fact that these invariants, which are a priori only stack valued, are in fact variety valued, so that taking Euler characteristic is legitimate, and produces integers.

If $X \rightarrow Y$ is a small resolution of a toric Gorenstein singularity, the calculation of motivic Donaldson–Thomas invariants has received by now a fairly comprehensive treatment (see [1], [35], [36]). Let $K^{\hat{\mu}}(\text{Var} / \text{Spec}(\mathbb{C}))$ be the ring of $\hat{\mu}$ -equivariant varieties, then there is a ring homomorphism $K^{\hat{\mu}}(\text{Var} / \text{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}] \rightarrow \mathbb{Z}[q^{1/2}]$, obtained by first taking the Hodge spectrum, a homomorphism to the ring of polynomials in two variables with fractional powers, and then specialising $u = v = q^{1/2}$. Furthermore, this is a retraction of rings, since there is a right inverse taking $q^{1/2}$ to $-\mathbb{L}^{1/2}$. The Donaldson–Thomas invariants that arise in the study of the above resolution $X \rightarrow Y$ all lie in the obviously very well-understood subring that is the image of this retract.

By contrast, the ring $K^{\hat{\mu}}(\mathrm{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}]$, as a whole, has a rich ring structure, with the product given by Looijenga's ‘exotic’ convolution product (see [33], [13], [29]), and a λ -ring structure studied in the paper [7], in which the first case to really utilise this extra structure is discussed - the calculation of the Donaldson–Thomas invariants for the one loop quiver with potential.

The present paper represents perhaps the first case where ‘natural’ Donaldson–Thomas invariants living in the interesting part of the ring $K^{\hat{\mu}}(\mathrm{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}]$ are discussed. Of course the question of naturalness here is subjective - we are appealing to the sensibilities of Algebraic Geometers, in that we consider an example that is manifestly a part of 3-dimensional Geometry, as opposed to the case of the one loop quiver with potential, which in the homogeneous case gives rise to the algebra $\mathbb{C}[x]/(x^d)$, which looks rather more like zero-dimensional geometry. Slightly more specifically, we consider the motivic Donaldson–Thomas invariants of (-2) curves, which are, for us, resolutions $Y_d \rightarrow X_d$ of singularities as defined in equation (1). In birational geometry and physics, these curves have a very long and rich history, see [38], [31] and [24] for example.

Our paper also seems to represent the first serious attempt to calculate Donaldson–Thomas invariants while keeping as true as possible to the framework of [29]. A side-effect of this approach is that some discussion of orientation data is necessitated. It is hoped that seeing this aspect of the story in action will help to demystify it a little. For the sake of those who would like to swap the (ever decreasingly) conjectural framework of [29] for the single very reasonable-looking conjecture of [7], we prove a slight variant of our main result at the end of the paper, avoiding all mention of orientation data, cyclic 3-Calabi–Yau categories and minimal potentials. In both cases we work with an algebraic model of the derived category of compactly supported coherent sheaves on Y_d , provided by considering modules over an algebra A_{Q_{-2}, W_d} , which is the free path algebra of the quiver in Figure 1, quotiented by some relations determined by the noncommutative derivatives of a potential W_d . Our main result is Theorem 5.4, which states that

$$\Phi_{Q_{-2}, W_d}([\mathcal{X}_{Q_{-2}, W_d}^{\mathrm{nilp}}]) = \mathrm{Sym} \left(\sum_{n \geq 0} \frac{\mathbb{L}^{-1/2}(1 - [\mu_{d+1}])}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} (\hat{e}_{(n, n+1)} + \hat{e}_{(n+1, n)}) + \sum_{n \geq 1} \frac{\mathbb{L}^{-1/2} + \mathbb{L}^{-3/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \hat{e}_{(n, n)} \right),$$

where the quantity on the left hand side is by definition the motivic generating series for nilpotent modules over A_{Q_{-2}, W_d} , which on the geometric side corresponds to counting coherent sheaves on the exceptional locus of $Y_d \rightarrow X_d$. It follows, by definition, that the motivic Donaldson–Thomas invariants Ω^{nilp} are given by $\Omega^{\mathrm{nilp}}(\mathbf{n}) = \mathbb{L}^{-1/2}(1 - [\mu_{d+1}])$ for $\mathbf{n} = (n, n+1)$ or $\mathbf{n} = (n+1, n)$, and $\mathbb{P}^1 \cdot \mathbb{L}^{-3/2}$ if $\mathbf{n} = (n, n)$. Here μ_{d+1} is considered as a μ_{d+1} -equivariant variety in the natural way, and so we have indeed produced motivic Donaldson–Thomas invariants with nontrivial monodromy, arising ‘in nature’ e.g. String Theory, and confirmed integrality, all the way up to the motivic level, for the Donaldson–Thomas invariants of (-2) -curves.

2. THE GEOMETRY OF (-2) -CURVES

In this paper we study the motivic Donaldson–Thomas invariants of local (-2) -curves, which are defined in the following way. We assume that $f : Y \rightarrow X$ is a resolution of a Gorenstein complex 3-fold singularity with exceptional curve $C \cong \mathbb{P}^1$, satisfying the condition that $f^*\omega_X \cong \omega_Y$, $\omega_Y \cdot C = 0$ and $N_{C|Y} \cong \mathcal{O}_C \oplus \mathcal{O}_C(-2)$ or $N_{C|Y} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Then (see [27]) we may assume that X is one of the singularities

$$(1) \quad X_d = \mathrm{Spec} \left(\mathbb{C}[x, y, z, w] / (x^2 + y^2 + (z + w^d)(z - w^d)) \right)$$

for $d \geq 1$, and Y is given by one of the two resolutions provided by blowing up along $0 = x = z \pm w^d$ (we denote by Y_d the blowup along $0 = x = z + w^d$, and by Y_d^+ the blowup along $0 = x = z - w^d$). The birational morphism $Y_d \rightarrow Y_d^+$ is the flop of this curve, and there is an equivalence of categories

$$(2) \quad \mathrm{D}^b(\mathrm{Coh}(Y_d)) \rightarrow \mathrm{D}^b(\mathrm{Coh}(Y_d^+))$$

with Fourier–Mukai kernel $\mathcal{O}_{Y_d \times_{X_d} Y_d^+}$. This is an example of a generalised spherical twist (see [42]). This equivalence is not given by an equivalence of the hearts of these two categories (even though they are

in fact equivalent, as there is an obvious isomorphism of schemes $Y_d \rightarrow Y_d^+$). As in [38] one defines the width¹ of C_d to be the length of the component of the moduli space of coherent sheaves on Y containing \mathcal{O}_{C_d} . Alternatively, one can define it as the dimension of the maximal commutative local Artinian algebra A such that one can embed $C_d \times \text{Spec}(A)$ as a subscheme of Y_d , with the fibre over the maximal ideal of A equal to the inclusion of C_d in Y_d . In fact one can show from the explicit description of X_d and Y_d that the maximal such A is just $\mathbb{C}[x]/(x^d)$ and the width of $C_d \subset Y_d$ is d .

For the purposes of this paper we will be interested in a derived equivalence that is different to that of equation (2). That is, we will be interested in a derived equivalence between the category of coherent sheaves on Y_d and the category of finitely generated right modules $\text{Mod-}A_{Q_{-2}, W_d}$ for a noncommutative algebra A_{Q_{-2}, W_d} . The approach to defining and studying Donaldson–Thomas invariants of categories of coherent sheaves is as initiated in [39], where the case of the ‘noncommutative conifold’ is considered, and indeed we will recover (motivic) Donaldson–Thomas invariants for the noncommutative conifold, as it is a special case of a (-2)-curve.²

The existence of the algebra A_{Q_{-2}, W_d} satisfying

$$(3) \quad \text{D}^b(\text{Mod-}A_{Q_{-2}, W_d}) \xrightarrow{\sim} \text{D}^b(\text{Coh}(Y_d))$$

is provided by the results of Van den Bergh [43]. It will help to have an explicit description of Y_d . It is covered by two coordinate patches $U_1 = \text{Spec}(\mathbb{C}[x, y_1, y_2])$ and $U_2 = \text{Spec}(\mathbb{C}[w, z_1, z_2])$, which are glued along

$$\begin{aligned} x &= w^{-1} \\ z_1 &= x^2 y_1 + x y_2^d \\ z_2 &= y_2. \end{aligned}$$

In the case of the conifold, after the change of coordinates $z'_1 = w z_1 - z_2$, $z'_2 = -(1+w)z_1 + z_2$, $y'_1 = y_1$, $y'_2 = (x+1)y_1 + y_2$, we recover the usual presentation of the resolved conifold as the total space of the bundle $\mathcal{O}_{C_1}(-1) \oplus \mathcal{O}_{C_1}(-1)$ over $C_1 \cong \mathbb{P}^1_{\mathbb{C}}$. We define $\mathcal{O}_{Y_d}(-n) := \mathcal{O}_{Y_d}(nD)$ where D is the divisor cut out by the equation $x = 0$ in the above coordinate patches. Then by Van den Bergh’s theorem, we have a derived equivalence as in (3) if we set $A_{Q_{-2}, W_d} = \text{End}_{Y_d}(E_d)$, where we define $E_d := \mathcal{O}_{Y_d} \oplus \mathcal{O}_{Y_d}(-1)$. We follow the convention of [25], representing morphisms between the two line bundles \mathcal{O}_{Y_d} and $\mathcal{O}_{Y_d}(-1)$ by elements of $\mathbb{C}[w, z_1, z_2]$ under the identifications $\Gamma(U_2, \mathcal{O}_{Y_d}) \cong \mathbb{C}[w, z_1, z_2] \cong \Gamma(U_2, \mathcal{O}_{Y_d}(-1))$. The endomorphism algebra can then be represented by the quiver algebra depicted in Figure 1. We have the relations

$$(4) \quad \begin{aligned} AX &= YA \\ BX &= YB \\ XC &= CY \\ XD &= DY \\ X^d &= CA - DB \\ Y^d &= AC - BD. \end{aligned}$$

It follows that A_{Q_{-2}, W_d} admits a superpotential description in the sense of [12], with quiver Q_{-2} given by the quiver of Figure 1 and superpotential given by

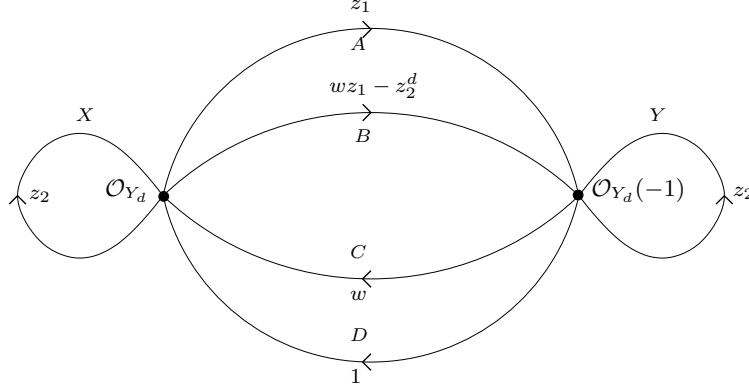
$$(5) \quad W_d = \frac{1}{d+1} X^{d+1} - \frac{1}{d+1} Y^{d+1} - XCA + XDB + YAC - YBD.$$

That is, we have an isomorphism

$$(6) \quad A_{Q_{-2}, W_d} \cong \mathbb{C} Q_{-2} / \langle \partial W_d / \partial E, E \in E(Q_{-2}) \rangle,$$

¹Not to be confused with the *length* of C_d , which is an entirely different invariant introduced by Kollár in [4] and used in the classification by Katz and Morrison [24] of irreducible small resolutions of Gorenstein 3-fold singularities.

²Note that the motivic Donaldson–Thomas invariants we obtain for the conifold differ from those of [35]; this is a result of a different choice of orientation data, in the terminology of [29].

FIGURE 1. Noncommutative resolution for a (-2) -curve

where for a general quiver Q and $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ given by a single loop, and $E \in E(Q)$ an arrow,

$$(7) \quad \partial W / \partial E := \sum_{aEb=W, a \text{ and } b \text{ paths in } Q} ba,$$

and for general W , $\partial W / \partial E$ is given by extending by linearity.

Definition 2.1. For a general quiver with potential Q, W , we define $A_{Q,W}$ in the same way as in equation (6). This is called the Jacobi algebra associated to the pair (Q, W) .

Remark 2.2. In the case of the conifold (i.e. if $d = 1$) there is a simpler presentation of the noncommutative resolution, considered in [39]. The quiver is given by Q_{con} , which is just Q_{-2} with the two loops X and Y removed. One sets $W_{\text{con}} = ACBD - ADBC$, and one can show directly that $A_{Q_{\text{con}}, W_{\text{con}}} \cong A_{Q_{-2}, W_1}$. Note that the relations (4) imply the relations given by the noncommutative derivatives of W_{con} , considered as a superpotential for Q_{-2} . As a result one may consider the morphisms assigned to X and Y for a A_{Q_{-2}, W_d} -module M as being together a morphism of a module M_{con} for $A_{Q_{\text{con}}, W_{\text{con}}}$, where M_{con} in turn is determined by the morphisms assigned to A, B, C and D by M , via the forgetful map.

3. NAIVE GROTHENDIECK RINGS OF MOTIVES

3.1. A λ -ring of motives. For \mathfrak{M} an Artin stack locally of finite type over \mathbb{C} we define $K_0(\text{St}^{\text{aff}}/\mathfrak{M})$ to be the Abelian group which is generated by isomorphism classes of morphisms $X \xrightarrow{f} \mathfrak{M}$ of finite type, with X a separated reduced stack over \mathbb{C} satisfying the condition that each of its \mathbb{C} -points has affine stabiliser, subject to the relations

$$[X \xrightarrow{f} \mathfrak{M}] \sim [Z \xrightarrow{f|_Z} \mathfrak{M}] + [X \setminus Z \xrightarrow{f|_{X \setminus Z}} \mathfrak{M}],$$

for $Z \subset X$ a closed substack of X . If $(\mathfrak{M}, \epsilon : \mathfrak{M} \times \mathfrak{M} \rightarrow \mathfrak{M}, 0 : \text{Spec}(\mathbb{C}) \rightarrow \mathfrak{M})$ is a (commutative) monoid in the category of Artin stacks over \mathbb{C} , then $K_0(\text{St}^{\text{aff}}/\mathfrak{M})$ acquires the structure of a (commutative) $K_0(\text{St}^{\text{aff}}/\text{Spec}(\mathbb{C}))$ -algebra, via convolution and the inclusion $[X \xrightarrow{f} \text{Spec}(\mathbb{C})] \mapsto [X \xrightarrow{0 \circ f} \mathfrak{M}]$. There are obvious G -equivariant versions $K_0^G(\text{St}^{\text{aff}}/\mathfrak{M})$ of the above groups and rings for G -equivariant stacks or monoids \mathfrak{M} , where we work with G -equivariant morphisms and assume that every point in X lies in a G -equivariant affine neighbourhood. Again we consider $K_0^G(\text{St}^{\text{aff}}/\mathfrak{M})$ as a $K_0(\text{St}^{\text{aff}}/\text{Spec}(\mathbb{C}))$ -algebra if \mathfrak{M} is a monoid in the category of locally finite type G -equivariant Artin stacks. For technical reasons it is better to work with a quotient $K^G(\text{St}^{\text{aff}}/\mathfrak{M})$ of $K_0^G(\text{St}^{\text{aff}}/\mathfrak{M})$ obtained by imposing the following relations.

- (1) We set $[X_1 \xrightarrow{f_1} \mathfrak{M}] \sim [X_2 \xrightarrow{f_2} \mathfrak{M}]$ if $[f_1^{-1}(U) \xrightarrow{f_1|_{f_1^{-1}(U)}} U] = [f_2^{-1}(U) \xrightarrow{f_2|_{f_2^{-1}(U)}} U]$ for all finite type open substacks $U \subset \mathfrak{M}$.

- (2) If $X' \xrightarrow{\pi} X$ is a G -equivariant vector bundle of rank r then $[X' \xrightarrow{f \circ \pi} \mathfrak{M}] \sim \mathbb{L}^r \cdot [X \xrightarrow{f \circ \pi} \mathfrak{M}]$ in $K^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$, where \mathbb{L} is the class of the affine line $\mathbb{A}_{\mathbb{C}}^1$ in $K_0(\mathrm{St}^{\mathrm{aff}}/\mathrm{Spec}(\mathbb{C}))$.

We define in the natural way the subgroup (or subring, if \mathfrak{M} is a monoid) $K^G(\mathrm{Var}/\mathfrak{M})$, spanned by classes $[X \rightarrow \mathfrak{M}]$ for X a G -equivariant variety over \mathbb{C} .

There is an equality $[\mathrm{Gl}_{\mathbb{C}}(n)] = \prod_{0 \leq i \leq n-1} (\mathbb{L}^n - \mathbb{L}^i)$ in $K(\mathrm{Var}/\mathrm{Spec}(\mathbb{C}))$.

Proposition 3.1 (cf. [10], Theorem 1.2). *The natural map $K^G(\mathrm{Var}/\mathfrak{M})[[\mathrm{Gl}_{\mathbb{C}}(n)]^{-1}, n \in \mathbb{N}] \rightarrow K^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$ is an isomorphism.*

For a morphism $h : \mathfrak{M} \rightarrow \mathfrak{T}$ of locally finite type Artin stacks we define $h^* : K^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{T}) \rightarrow K^G(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$ via the pullback (if h is representable, this is $(K(\mathrm{Var}/\mathrm{Spec}(\mathbb{C}))[[\mathrm{Gl}_{\mathbb{C}}(n)]^{-1}, n \in \mathbb{N}] \otimes_{K(\mathrm{Var}/\mathrm{Spec}(\mathbb{C}))} -)$ applied to the analogous map for varieties), and we define $\int_h : K^G(\mathrm{Var}/\mathfrak{M}) \rightarrow K^G(\mathrm{Var}/\mathfrak{T})$ via composition with h , if h is of finite type. For $j : \mathfrak{M}' \hookrightarrow \mathfrak{M}$ an inclusion of a finite type substack we write $\int_{\mathfrak{M}'} := \int_h \circ j^*$, where $h : \mathfrak{M}' \rightarrow \mathrm{Spec}(\mathbb{C})$ is the structure morphism.

We will briefly recall the framework of [7]. Let $(\mathfrak{M}, \epsilon, 0)$ be a monoid in the category of Artin stacks, locally of finite type. We will be interested in the group $K^{\mathbb{G}_m, n}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}_{\mathfrak{M}}^1)$, the naive Grothendieck group of \mathbb{G}_m -equivariant stacks over $\mathbb{A}_{\mathfrak{M}}^1 := \mathbb{A}_{\mathbb{C}}^1 \times \mathfrak{M}$, with this stack given the \mathbb{G}_m -action that is trivial on \mathfrak{M} and acts with weight n on $\mathbb{A}_{\mathbb{C}}^1$. The \mathbb{G}_m -equivariant projection map $p : \mathbb{A}_{\mathbb{C}}^1 \times \mathfrak{M} \rightarrow \mathfrak{M}$ induces a map $p^* : K^{\mathbb{G}_m}(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M}) \rightarrow K^{\mathbb{G}_m, n}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}_{\mathfrak{M}}^1)$ and we denote by \mathcal{I}_n the image of this map. We give \mathfrak{M} the trivial μ_n -action, where μ_n denotes the n -th roots of unity in \mathbb{C}^* . The map $K^{\mu_n}(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M}) \rightarrow K^{\mathbb{G}_m, n}(\mathrm{St}^{\mathrm{aff}}/\mathbb{A}_{\mathfrak{M}}^1)/\mathcal{I}_n$ given by $[Y \xrightarrow{f} \mathfrak{M}] \mapsto [Y \times_{\mu_n} \mathbb{G}_m \xrightarrow{(y, z) \mapsto (z^n, f(y))} \mathbb{A}_{\mathbb{C}}^1 \times \mathfrak{M}]$ is an isomorphism. For each $a \geq 1$ there is a natural morphism $\mu_{an} \rightarrow \mu_n$ sending z to z^a , and this induces an embedding $K^{\mu_n}(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M}) \rightarrow K^{\mu_{an}}(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})$, and we define $K^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})[\mathbb{L}^{-1/2}]$ to be the group obtained by taking the limit of these embeddings, and then adding a formal inverse to a formal square root of \mathbb{L} .

Definition 3.2. *Given a ring R , always assumed to be commutative, a pre λ -ring structure on R is given by a map $\sigma : R \rightarrow R[[T]]$ satisfying*

- $\sigma(0) = 1$
- $\sigma(a) = 1 + aT$ modulo $T^2 \cdot R[[T]]$
- $\sigma(a + b) = \sigma(a)\sigma(b)$.

One defines the operations $\sigma^n(r)$ via $\sigma(r) = \sum_{i \geq 0} \sigma^i(r)T^i$, and we define $\mathrm{Sym}(r) = \sum_{i \geq 0} \sigma^i(r)$ when this infinite sum exists.³ Finally, if R is a pre λ -ring we define a pre λ -ring structure on $R[[X]]$ by setting $\sigma^n(r \cdot X^i) := \sigma^n(r) \cdot X^{i \cdot n}$.

Using the above notation, we have $\mathrm{Sym}(\sum_{i \geq 1} a_i \cdot X^i) = \sum_{\pi} \prod_i (\sigma^{\pi(i)}(a_i)) X^{i \cdot \pi(i)}$, where the sum is over all partitions π , and we denote by $\pi(i)$ the number of parts of π of size i .

Proposition 3.3. *Let $(\mathfrak{M}, \epsilon, 0)$ be a commutative monoid in the category of varieties over \mathbb{C} . Denote by $+: \mathbb{A}_{\mathfrak{M}}^1 \times \mathbb{A}_{\mathfrak{M}}^1 \rightarrow \mathbb{A}_{\mathfrak{M}}^1$ the map $((z_1, x_1), (z_2, x_2)) \mapsto (z_1 + z_2, \epsilon(x_1, x_2))$ making $\mathbb{A}_{\mathfrak{M}}^1$ into a commutative monoid. The Abelian group $K^{\mathbb{G}_m, n}(\mathrm{Var}/\mathbb{A}_{\mathfrak{M}}^1)$ has the structure of a pre λ -ring, if we set*

$$[X_1 \xrightarrow{f_1} \mathbb{A}_{\mathfrak{M}}^1] \cdot [X_2 \xrightarrow{f_2} \mathbb{A}_{\mathfrak{M}}^1] = [X_1 \times X_2 \xrightarrow{f_1 \times f_2} \mathbb{A}_{\mathfrak{M}}^1 \times \mathbb{A}_{\mathfrak{M}}^1 \xrightarrow{+} \mathbb{A}_{\mathfrak{M}}^1]$$

and

$$\sigma^n([X \xrightarrow{f} \mathbb{A}_{\mathfrak{M}}^1]) = [\mathrm{Sym}^n X \xrightarrow{\mathrm{Sym}^n f} \mathrm{Sym}^n \mathbb{A}_{\mathfrak{M}}^1 \xrightarrow{+} \mathbb{A}_{\mathfrak{M}}^1],$$

for varieties X, X_1, X_2 . Furthermore, this induces a pre λ -ring structure on the quotient $K^{\mu_n}(\mathrm{Var}/\mathfrak{M})$, which is preserved by the embeddings $K^{\mu_n}(\mathrm{Var}/\mathfrak{M}) \rightarrow K^{\mu_{an}}(\mathrm{Var}/\mathfrak{M})$, giving rise to a pre λ -ring structure on $K^{\hat{\mu}}(\mathrm{Var}/\mathfrak{M})$.

Remark 3.4. There is a unique extension of this pre λ -ring structure to $K^{\hat{\mu}}(\mathrm{St}^{\mathrm{aff}}/\mathfrak{M})[\mathbb{L}^{-1/2}]$, using Proposition 3.1. This we may describe as follows. Given a symbol in $K^{\hat{\mu}}(\mathrm{Var}/\mathfrak{M})[\mathbb{L}^{-1/2}, [\mathrm{Gl}_{\mathbb{C}}(n)]^{-1}, n \in \mathbb{N}]$ one obtains a non-unique element P of $K^{\hat{\mu}}(\mathrm{Var}/\mathfrak{M})[[u]][u^{-1}]$ after substituting instances of $[1 - \mathbb{L}^n]^{-t}$

³This will be the case for $r \in F^1 R$ if R is a complete filtered ring with filtration $F^* R$ such that $\sigma^i(F^j R) \subset F^{i \cdot j} R$ for all $i, j \in \mathbb{N}$.

out for their power series expansions and then substituting⁴ $\mathbb{L}^{1/2} \mapsto -u$. It is not hard to verify that the set of formal power series obtained in this way is closed under taking σ^i for each i if we extend σ^i to the ring of Laurent series by $\sigma^i(au^j) = \sigma^i(a)u^{i \cdot j}$. One may then take $\sigma^i(P)$, followed by the substitution $u \mapsto -\mathbb{L}^{1/2}$, to arrive at a (unique) element of $K^{\hat{\mu}}(\text{Var}/\mathfrak{M})[\mathbb{L}^{-1/2}, [\text{Gl}_{\mathbb{C}}(n)]^{-1}, n \in \mathbb{N}]$.

Definition 3.5. A power structure on a ring R is a map $(1 + T \cdot R[[T]]) \times R \rightarrow (1 + T \cdot R[[T]])$, written $(A(T), m) \mapsto A(T)^m$, satisfying

- $A(T)^0 = 1$,
- $A(T)^1 = A(T)$,
- $(A(T) \cdot B(T))^m = A(T)^m \cdot B(T)^m$,
- $A(T)^{m+n} = A(T)^m \cdot A(T)^n$, $A(T)^{m \cdot n} = (A(T)^m)^n$,
- $(1 + T)^m$ is equal to $1 + m \cdot T$ modulo $T^2 \cdot R[[T]]$,
- $A(T^a)^m = A(T)_{T \rightarrow T^a}^m$.

We assume all power structures are continuous with respect to the T -adic topology on $R[[T]]$.

Given a power series $A(T) \in 1 + T \cdot R[[T]]$ we may write $A(T)$ uniquely as an infinite product $A(T) = \prod_{n \geq 1} (1 - T^n)^{-a_n}$. It follows that there is a one to one correspondence between continuous power structures and pre λ -ring structures: given a pre λ -ring structure we write $A(T)^m = \text{Sym}(\sum_{n \geq 1} m a_n \cdot T^n)$, and given a finitely determined power structure on R we may write $\sigma(m) = (1 - T)^{-m}$. For R a ring, and R' a quotient ring of R , power structures on R descending to power structures on R' are exactly the power structures such that the associated pre λ -ring structure descends to R' .

Proposition 3.6. The power structure on $K^{\mathbb{G}_m, n}(\text{Var}/\mathbb{A}_{\mathfrak{M}}^1)$ inducing the pre λ -ring structure of Proposition 3.3 is defined by

$$(8) \quad \left(\sum_{n \geq 0} [A_n \rightarrow \mathbb{A}_{\mathfrak{M}}^1] \cdot T^n \right)^{[B \xrightarrow{g} \mathbb{A}_{\mathfrak{M}}^1]} := \sum_{\pi} \left[\left(\prod_i (B^{\pi_i} \times A_i^{\pi_i}) / S_{\pi_i} \right) \setminus \Delta \xrightarrow{+ \circ \prod_i g^{\pi_i} \times f_i^{\pi_i}} \mathbb{A}_{\mathfrak{M}}^1 \right] \cdot T^{\sum_i i \pi_i},$$

where the sum is over all functions $\pi : \mathbb{N} \rightarrow \mathbb{N}$ with finite support and Δ is the preimage of the big diagonal in $\prod_i B^{\pi_i} / S_{\pi_i}$ with respect to the obvious projection. This power structure descends to $K^{\mu_n}(\text{Var}/\mathfrak{M})$ and is preserved by the embeddings $K^{\mathbb{G}_m, n}(\text{Var}/\mathfrak{M}) \rightarrow K^{\mathbb{G}_m, an}(\text{Var}/\mathfrak{M})$ and induces a power structure on $K^{\hat{\mu}}(\text{Var}/\mathfrak{M})$.

Given π one should think of $\prod_i (B^{\pi_i} \times A_i^{\pi_i}) / S_{\pi_i}$ as being the configuration space of pairs (K, ϕ) , where K is a finite subset of B of cardinality $\sum_i \pi_i$ and $\phi : K \rightarrow \prod_i A_i$ is a map sending π_i points to A_i .

Proof. The given power structure on $K^{\mathbb{G}_m, n}(\text{St}^{\text{aff}}/\mathfrak{M})$ can be checked to be a power structure inducing the given pre λ -ring structure as in [14]. The statements regarding the preservation of power structures under embeddings and their descent to the quotient are true due to the correspondence between power structures and pre λ -rings, and the truth of the corresponding statements on the side of pre λ -rings - this is just Proposition 3.3 again. \square

Remark 3.7. This power structure extends to $K^{\mathbb{G}_m, n}(\text{St}^{\text{aff}}/\mathbb{A}_{\mathfrak{M}}^1)$ inducing a power structure on $K^{\hat{\mu}}(\text{St}^{\text{aff}}/\mathfrak{M})$ which corresponds to the pre λ -ring structure discussed in Remark 3.4. In order to do this, we have to replace $[B \xrightarrow{g} \mathbb{A}_{\mathfrak{M}}^1]$ and $[A_i \xrightarrow{f_i} \mathbb{A}_{\mathfrak{M}}^1]$ with formal power series $B = \sum_j [B_j \xrightarrow{g_j} \mathbb{A}_{\mathfrak{M}}^1] u^j$ and $A_i = \sum_k [A_{ik} \xrightarrow{f_{ik}} \mathbb{A}_{\mathfrak{M}}^1] u^k$ in u with coefficients in $K^{\mathbb{G}_m, n}(\text{Var}/\mathbb{A}_{\mathfrak{M}}^1)$. To get the correct formula, we should think of these series as being the motives of $\prod_j B_j \rightarrow \mathbb{A}_{\mathfrak{M}}^1$ resp. $\prod_k A_{ik} \rightarrow \mathbb{A}_{\mathfrak{M}}^1$. The configuration spaces decompose accordingly. If π_{ijk} denotes the number of points in B_j mapped into A_{ik} , then the correct form of the right hand side of the formula in Proposition 8 is

$$\sum_{\pi} \left[\left(\prod_{i,j,k} (B_j^{\pi_{ijk}} \times A_{ik}^{\pi_{ijk}}) / S_{\pi_{ijk}} \right) \setminus \Delta \xrightarrow{+ \circ \prod_{i,j,k} g_j^{\pi_{ijk}} \times f_{ik}^{\pi_{ijk}}} \mathbb{A}_{\mathfrak{M}}^1 \right] \cdot u^{\sum_{i,j,k} (j+k) \pi_{ijk}} T^{\sum_{i,j,k} i \pi_{ijk}},$$

⁴Note we explicitly choose $\mathbb{L}^{1/2}$ not to be a line element.

where the sum is now taken over all functions $\pi : \mathbb{N}^3 \rightarrow \mathbb{N}$ with compact support and Δ denotes the preimage of the big diagonal in $\prod_{i,j,k} B_j^{\pi_{ijk}} / S_{\pi_{ijk}}$ with respect to the obvious projection.

3.2. Motivic Hall algebras. We recall the definition of the motivic Hall algebra for the stack of finite dimensional $A_{Q,W}$ -modules, for $A_{Q,W}$ a Jacobi algebra as in Definition 2.1. For Q a finite quiver and $\mathbf{n} \in \mathbb{N}^{V(Q)}$ a dimension vector, we define the moduli stack

$$\mathcal{Y}_{Q,\mathbf{n}} := \prod_{a \in E(Q)} \operatorname{Hom} \left(\mathbb{C}^{\mathbf{n}(t(a))}, \mathbb{C}^{\mathbf{n}(s(a))} \right) / \prod_{i \in V(Q)} \operatorname{Gl}_{\mathbb{C}}(\mathbf{n}(i)),$$

where $s(a)$ is the source of the arrow a and $t(a)$ is the target⁵, and $\operatorname{Gl}_{\mathbb{C}}(\mathbf{n}(i))$ acts by change of basis of $\mathbb{C}^{\mathbf{n}(i)}$. We define $\mathcal{Y}_Q := \coprod_{\mathbf{n} \in \mathbb{N}^{V(Q)}} \mathcal{Y}_{Q,\mathbf{n}}$. If $W \in \mathbb{C}Q / [\mathbb{C}Q, \mathbb{C}Q]$ is a superpotential we define $\mathcal{X}_{Q,W,\mathbf{n}}$ to be the Zariski closed subscheme of $\mathcal{Y}_{Q,\mathbf{n}}$ cut out by the matrix valued equations given by the noncommutative partial differentials (as defined by equation (7) and the line following it) of W , and $\mathcal{X}_{Q,W} := \coprod_{\mathbf{n} \in \mathbb{N}^{(Q)_0}} \mathcal{X}_{Q,W,\mathbf{n}}$ to be the moduli stack of finite dimensional modules for $A_{Q,W}$, the Jacobi algebra for (Q, W) . Denote by $\mathcal{X}_{Q,W}^{\text{nilp}}$ and $\mathcal{Y}_Q^{\text{nilp}}$ the moduli stacks of finite dimensional nilpotent right modules for $A_{Q,W}$ and $\mathbb{C}Q$ respectively.

The Abelian groups $K(\operatorname{St}^{\text{aff}} / \mathcal{Y}_Q)$, $K(\operatorname{St}^{\text{aff}} / \mathcal{Y}_Q^{\text{nilp}})$, $K(\operatorname{St}^{\text{aff}} / \mathcal{X}_{Q,W})$ and $K(\operatorname{St}^{\text{aff}} / \mathcal{X}_{Q,W}^{\text{nilp}})$ carry a Hall algebra product for which the comprehensive reference is the series of papers by Dominic Joyce (see [15], [16], [17], [20] or also Bridgeland's summary [3]). For completeness we recall the definition. We fix our attention on $K(\operatorname{St}^{\text{aff}} / \mathcal{Y}_Q)$ for now. Let $[X_i \xrightarrow{f_i} \mathcal{Y}_Q]$ be two effective classes, for $i = 0, 1$. Since the product is linear, and $K(\operatorname{St}^{\text{aff}} / \mathcal{Y}_Q)$ is isomorphic to the inverse limit of the quotients $\mathfrak{Q}_t := K(\operatorname{St}^{\text{aff}} / \mathcal{Y}_Q) / K(\operatorname{St}^{\text{aff}} / \coprod_{\mathbf{n} \in \mathbb{N}^{(Q)_0} \text{ such that } |\mathbf{n}| \geq t} \mathcal{Y}_{Q,\mathbf{n}})$ we may assume that each morphism f_i factors through an inclusion $\mathcal{Y}_{Q,\mathbf{n}_i} \hookrightarrow \mathcal{Y}_Q$. For $a, b \in \mathbb{N}$, denote by $\operatorname{Gl}_{\mathbb{C}}(a, b)$ the Borel subgroup of $\operatorname{Gl}_{\mathbb{C}}(a + b)$ preserving the standard flag $0 = \mathbb{C}^0 \subset \mathbb{C}^a \subset \mathbb{C}^{a+b}$. Let

$$Y_{Q,\mathbf{n}_0,\mathbf{n}_1} \subset \prod_{a \in E(Q)} \operatorname{Hom} \left(\mathbb{C}^{\mathbf{n}_0(t(a)) + \mathbf{n}_1(t(a))}, \mathbb{C}^{\mathbf{n}_0(s(a)) + \mathbf{n}_1(s(a))} \right)$$

be the subspace such that the points of $Y_{Q,\mathbf{n}_0,\mathbf{n}_1}$ correspond to linear maps preserving the standard flag

$$0 = \bigoplus_{i \in V(Q)} \mathbb{C}^0 \subset \bigoplus_{i \in V(Q)} \mathbb{C}^{\mathbf{n}_0(i)} \subset \bigoplus_{i \in V(Q)} \mathbb{C}^{\mathbf{n}_0(i) + \mathbf{n}_1(i)},$$

and let $\mathcal{Y}_{Q,\mathbf{n}_0,\mathbf{n}_1} = Y_{Q,\mathbf{n}_0,\mathbf{n}_1} / \prod_{i \in V(Q)} \operatorname{Gl}_{\mathbb{C}}(\mathbf{n}_0(i), \mathbf{n}_1(i))$. Then there are three natural morphisms of stacks

$$\begin{aligned} \pi_1 : \mathcal{Y}_{Q,\mathbf{n}_0,\mathbf{n}_1} &\rightarrow \mathcal{Y}_{Q,\mathbf{n}_0} \\ \pi_2 : \mathcal{Y}_{Q,\mathbf{n}_0,\mathbf{n}_1} &\rightarrow \mathcal{Y}_{Q,\mathbf{n}_0 + \mathbf{n}_1} \\ \pi_3 : \mathcal{Y}_{Q,\mathbf{n}_0,\mathbf{n}_1} &\rightarrow \mathcal{Y}_{Q,\mathbf{n}_1}, \end{aligned}$$

and we define $[X_0 \xrightarrow{f_0} \mathcal{Y}_Q] \star [X_1 \xrightarrow{f_1} \mathcal{Y}_Q]$ to be the composition given by the top row of the following commutative diagram

$$\begin{array}{ccccc} X_2 & \longrightarrow & \mathcal{Y}_{Q,\mathbf{n}_0,\mathbf{n}_1} & \xrightarrow{\pi_2} & \mathcal{Y}_{Q,\mathbf{n}_0 + \mathbf{n}_1} \hookrightarrow \mathcal{Y}_Q \\ \downarrow & & \downarrow \pi_1 \times \pi_3 & & \\ X_0 \times X_1 & \xrightarrow{f_0 \times f_1} & \mathcal{Y}_{Q,\mathbf{n}_0} \times \mathcal{Y}_{Q,\mathbf{n}_1}, & & \end{array}$$

where the leftmost square is the pullback. This gives a well defined product on each of the quotients \mathfrak{Q}_t , and so it gives a well defined product on $K(\operatorname{St}^{\text{aff}} / \mathcal{Y}_Q)$. It is easy to see that under the Hall algebra product each of the groups $K(\operatorname{St}^{\text{aff}} / \mathcal{Y}_Q^{\text{nilp}})$, $K(\operatorname{St}^{\text{aff}} / \mathcal{X}_{Q,W})$ and $K(\operatorname{St}^{\text{aff}} / \mathcal{X}_{Q,W}^{\text{nilp}})$ is a subalgebra.

⁵It's generally better to work with right modules, which is why our homomorphisms go from the vector space labelled by the target of the arrow to the space labelled by the source.

3.3. Motivic vanishing cycles. Let X be a smooth scheme over \mathbb{C} and let $f : X \rightarrow \mathbb{A}_{\mathbb{C}}^1$ be a regular map. One defines $\mathcal{L}_n(X)$, the space of arcs in X of length n , to be the scheme representing the functor $Y \mapsto \text{Hom}_{\text{Sch}}(Y \times_{\mathbb{C}} \text{Spec}(\mathbb{C}[t]/t^{n+1}), X)$. Via the natural inclusion $\text{Spec}(\mathbb{C}[t]/t) \rightarrow \text{Spec}(\mathbb{C}[t]/t^{n+1})$ there is a map of schemes $p_n : \mathcal{L}_n(X) \rightarrow X$. We write $\mathcal{L}_n(X)|_{X_0} = p_n^{-1}f^{-1}(0)$. There is a natural morphism $f_* : \mathcal{L}_n(X) \rightarrow \mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^1)$. An arc in $\mathbb{A}_{\mathbb{C}}^1$ is given by a polynomial $a_0 + \dots + a_nt^n$, and so $\mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^1) \cong \mathbb{A}_{\mathbb{C}}^{n+1}$ and the composition of f_* with the projection $\pi : \mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^1) \rightarrow \mathbb{A}_{\mathbb{C}}^1$ given by $a_0 + \dots + a_nt^n \mapsto a_n$ makes $\mathcal{L}_n(X)$ into a scheme over $\mathbb{A}_{\mathbb{C}}^1$. Moreover there is a \mathbb{G}_m -action on $\mathcal{L}_n(X)$ given by rescaling the coordinate t of $\mathbb{C}[t]/t^{n+1}$, and $[\mathcal{L}_n(X)|_{X_0} \xrightarrow{\pi \circ f_* \times p_n} \mathbb{A}_{X_0}^1] \in K^{\mathbb{G}_m, n}(\text{Var}/\mathbb{A}_{X_0}^1)$. We consider the expression

$$Z_f^{\text{eq}}(T) := \sum_{n \geq 1} \mathbb{L}^{-(n+1)\dim(X)/2} \cdot [\mathcal{L}_n(X)|_{X_0} \xrightarrow{p_n \times \pi \circ f_*} \mathbb{A}_{X_0}^1] T^n$$

as a formal power series with coefficients in $K^{\hat{\mu}}(\text{Var}/X_0)[\mathbb{L}^{-1/2}]$. In general (see [8]) it makes sense to evaluate this function at infinity, and one defines $\phi_f = -Z_f^{\text{eq}}(\infty) \in K^{\hat{\mu}}(\text{Var}/X_0)[\mathbb{L}^{-1/2}]$, the motivic vanishing cycle⁶ of f .

The motivic vanishing cycle has the property that if $g : X_1 \rightarrow X_2$ is a smooth morphism of smooth schemes, and $f : X_2 \rightarrow \mathbb{A}_{\mathbb{C}}^1$ is a regular function, then $\phi_{f \circ g} = \mathbb{L}^{-\dim(g)/2} \cdot g^* \phi_f$. Given an Artin stack \mathcal{Z} that is a quotient stack $Z/\text{GL}_{\mathbb{C}}(m)$ for smooth connected Z , and $f : \mathcal{Z} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ a function, one defines⁷ $\phi_f = \mathbb{L}^{m^2/2} \cdot [\text{BGL}_{\mathbb{C}}(m)] \cdot \pi_* \phi_{f \circ \pi} \in K^{\hat{\mu}}(\text{St}^{\text{aff}}/\mathcal{Z})$, where $\pi : Z \rightarrow \mathcal{Z}$ is the projection.

In studying 3-dimensional Calabi–Yau categories, one is often faced with the following situation, which necessitates the use of a *relative* version of motivic vanishing cycles. Firstly, let X be a finite type scheme, carrying a constructible vector bundle V , with a function $f : \text{Tot}(V) \rightarrow \mathbb{C}$ vanishing on the zero fibre. By constructible vector bundle, we mean that there is a finite decomposition of X into locally closed subschemes $X = \coprod X_i$, and a vector bundle V_i on each of the X_i (we don’t assume these vector bundles are of the same dimension). By a function on such an object we mean a function on each of the V_i , possibly after further decomposition. In full generality, one should consider formal functions on V , by which we mean a function on the formal neighbourhood of the zero section of each of the V_i . We would like to define a motivic vanishing cycle for such an object. This we do by defining $\mathcal{L}_n(V)$ to be the space of those arcs in $\text{Tot}(V)$ that restrict to a single fibre of the projection $\pi : V \rightarrow X$, and we define $\mathcal{L}_n(V)|_X = p_n^{-1}(X)$ as before. Finally, define

$$Z_f^{\text{eq}}(T) := \sum_{n \geq 1} \mathbb{L}^{-(n+1)\dim(V)/2} \cdot [\mathcal{L}_n(V)|_X \xrightarrow{p_n \times \pi \circ f_*} \mathbb{A}_X^1] T^n \in K^{\hat{\mu}}(\text{Var}/X)[\mathbb{L}^{-1/2}],$$

where $\dim(V)$ is the dimension of the vector bundle (not the dimension of the total space). Up to constructible decomposition, we may assume that X is smooth and connected and V is an honest vector bundle. Then we may find an embedded resolution of singularities over the generic point (see [5]), and use the usual machinery (as in [8] - we consider here only algebraic functions on V , not formal ones) to show that, at least over some open $U \subset X$, this formal power series can be evaluated at infinity. Then, using Noetherian induction, we have obtained a well defined motivic vanishing cycle for f , which we denote by ϕ_f^{rel} .

4. MOTIVIC DONALDSON–THOMAS THEORY

4.1. Three-dimensional Calabi–Yau categories. We recall the essential ingredients of the theory of motivic Donaldson–Thomas invariants from [29]. We will begin with the data that one feeds into this machine. One starts with \mathcal{C} , a 3-Calabi–Yau category. By a 3-Calabi–Yau category \mathcal{C} we mean a set of objects $\text{ob}(\mathcal{C})$, between any two objects $x_i, x_j \in \text{ob}(\mathcal{C})$ a \mathbb{Z} -graded vector space $\text{Hom}_{\mathcal{C}}(x_i, x_j)$, and a countable collection of operations

$$b_{\mathcal{C}, n} : \text{Hom}_{\mathcal{C}}(x_{n-1}, x_n)[1] \otimes \dots \otimes \text{Hom}_{\mathcal{C}}(x_0, x_1)[1] \rightarrow \text{Hom}_{\mathcal{C}}(x_0, x_n)[1]$$

⁶This definition differs by a factor of $(-\mathbb{L}^{1/2})^{\dim(X)}$ from the original definition of Denef and Loeser. It makes the motivic weights appearing in Donaldson–Thomas theory simpler.

⁷Note that, by relation (2), $[\text{BGL}_{\mathbb{C}}(m)] = [\text{GL}_{\mathbb{C}}(m)]^{-1} \in K(\text{St}^{\text{aff}}/\text{Spec}(\mathbb{C}))$

of degree 1, satisfying the condition

$$\sum_{\alpha+\beta+\gamma=n} b_{\mathcal{C},\alpha+1+\gamma} \circ (\mathbf{1}^{\otimes\alpha} \otimes b_{\mathcal{C},\beta} \otimes \mathbf{1}^{\otimes\gamma}) = 0.$$

See [32] for a comprehensive guide to A_∞ -categories, or [23] for a similarly comprehensive guide to cyclic A_∞ -categories, or [26] for a gentle and concise reference for most of what follows. All these ideas are also covered in the notes [28]. The 3-Calabi–Yau condition consists of the extra data of a skewsymmetric nondegenerate bracket $\langle \bullet, \bullet \rangle_{\mathcal{C}} : \text{Hom}_{\mathcal{C}}(x_i, x_j)[1] \otimes \text{Hom}_{\mathcal{C}}(x_j, x_i)[1] \rightarrow \mathbb{C}$ of degree -1, such that the functions $W_{\mathcal{C},n} := \langle b_{\mathcal{C},n-1}(\bullet, \dots, \bullet), \bullet \rangle$ are cyclically symmetric. One defines $W_{\mathcal{C}}(z) = \sum_{i \geq 2} W_{\mathcal{C},i}(z, \dots, z)/n$, a formal function on $\text{Hom}_{\mathcal{C}}^1(x_i, x_i)$ for each $x_i \in \text{ob}(\mathcal{C})$.

In our case this category is given by $\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))$, where Q_{-2} and W_d are the quiver with potential from Section 2. We will briefly recall the construction and the required properties of this category, but firstly we should motivate the introduction of this category into a paper that is about geometry. Let us start with the Ginzburg dg category $\Gamma(Q, W)$ associated to a quiver Q with potential W . It is constructed as follows. The quiver Q is given by the data of a bimodule S for the semisimple ring $R := \mathbb{C}^{V(Q)}$, where we set

$$\dim(e_i \cdot S \cdot e_j) := \# \text{ of arrows from } j \text{ to } i.$$

The objects of the category $\Gamma(Q, W)$ are just the vertices of the quiver, i.e. $\text{ob}(\Gamma(Q, W)) := V(Q)$, and for two vertices x_i, x_j we put

$$\text{Hom}_{\Gamma(Q, W)}(x_i, x_j) = e_j \cdot T_R(R[2] \oplus S^*[1] \oplus S) \cdot e_i,$$

where $e_i, e_j \in R$ are the idempotent elements corresponding to x_i resp. x_j . Moreover, $T_R(M)$ denotes the completed tensor R -bialgebra of a graded R -bimodule M . Composition in the category $\Gamma(Q, W)$ is given by the tensor product. In addition to this, we define a differential d of degree one on $T_R(R[2] \oplus S^*[1] \oplus S)$ such that $d(e_i[2]) = \sum_{a_k: x_i \rightarrow x_j} a_k^* a_k - \sum_{a_k: x_j \rightarrow x_i} a_k a_k^*$, $d(a_k^*[1]) = \partial W / \partial a_k$ and $d(a_k) = 0$, where a_k (resp. a_k^*) runs through a (dual) basis of the vector space $e_j S e_i$ (resp. $e_i S^* e_j = (e_j S e_i)^*$). This makes $\Gamma(Q, W)$ into a dg and hence into an A_∞ -category. There is a natural equivalence of categories between the full subcategory of degree zero finite-dimensional modules over $\Gamma(Q, W)$ and nilpotent finite dimensional modules over the Jacobi algebra $A_{Q, W}$. The dg category $\mathbf{Mod}\text{-}\Gamma(Q, W)$ of dg modules over $\Gamma(Q, W)$ is not 3-Calabi-Yau but the dg subcategory $\text{D}_{\text{cf}}^b(\Gamma(Q, W)) \subset \text{D}_{\text{cf}}(\Gamma(Q, W)) \cong \text{Fun}_{\text{cf}}^{\text{dg}}(\Gamma(Q, W), \text{Vect}_{\mathbb{C}}^{\mathbb{Z}})$ consisting of (fibrant and) cofibrant⁸ right $\Gamma(Q, W)$ dg modules with finite dimensional total cohomology is naturally quasi-equivalent to a 3-Calabi-Yau category. To get a better picture, we mention that $\Gamma(Q, W)$ is actually the Koszul dual (see for example [34] for this notion) of another A_∞ -category $\mathcal{D}(Q, W)$ which is manifestly 3-Calabi-Yau - this is what motivates our discussion of $\mathcal{D}(Q, W)$.

As before, the objects of the category $\mathcal{D}(Q, W)$ are just the vertices of the quiver, i.e. $\text{ob}(\mathcal{D}(Q, W)) := V(Q)$. The homomorphism spaces between these objects are graded vector spaces concentrated in degrees between zero and three. One sets

$$(9) \quad \text{Hom}_{\mathcal{D}(Q, W)}^n(x_i, x_j) := \begin{cases} \mathbb{C}^{\delta_{ij}} & \text{if } n = 0 \\ (e_i \cdot S \cdot e_j)^* & \text{if } n = 1 \\ (e_j \cdot S \cdot e_i) & \text{if } n = 2 \\ (\mathbb{C}^*)^{\delta_{ij}} & \text{if } n = 3, \end{cases}$$

where δ_{ij} is the Kronecker delta function. The A_∞ operations on this category are given by firstly setting the natural generator 1_i of $\text{Hom}_{\mathcal{D}(Q, W)}^0(x_i, x_i)$ to be a strict unit for every $i \in Q_0$. This means that $b_2(f, 1_i) = f$ and⁹ $b_2(1_i, g) = -g$ for all $f \in \text{Hom}_{\mathcal{D}(Q, W)}(x_i, x_j)$ and $g \in \text{Hom}_{\mathcal{D}(Q, W)}(x_j, x_i)$, and any insertion of 1_i into b_n for any $n \geq 3$ results in the zero function. We let $b_{\mathcal{D}(Q, W), 2}(\theta, z) = -\theta(z)1_j^*$ with

⁸Such modules are given by direct sums of the projective modules x_i , where these objects are considered as right $\Gamma(Q, W)$ modules via the Yoneda embedding

⁹The strange sign here is the price we pay for considering the maps $b_n : \text{Hom}_{\mathcal{C}}(x_{n-1}, x_n)[1] \otimes \dots \otimes \text{Hom}_{\mathcal{C}}(x_0, x_1)[1] \rightarrow \text{Hom}_{\mathcal{C}}(x_0, x_n)[1]$ instead of $m_n : \text{Hom}_{\mathcal{C}}(x_{n-1}, x_n) \otimes \dots \otimes \text{Hom}_{\mathcal{C}}(x_0, x_1) \rightarrow \text{Hom}_{\mathcal{C}}(x_0, x_n)$. The payoff is that there are a lot fewer signs overall.

$1_j^* \in \text{Hom}_{\mathbb{C}}^3(x_j, x_j)$ being the dual basis of 1_j , and $b_{\mathcal{D}(Q,W),2}(z, \theta) = \theta(z)1_i^*$ for any $\theta \in \text{Hom}_{\mathcal{D}(Q,W)}(x_i, x_j)$ and $z \in \text{Hom}_{\mathcal{D}(Q,W)}(x_j, x_i)$. Then (for degree reasons) all that is left is to define the degree zero operations

$$b_{\mathcal{D}(Q,W),m} : \text{Hom}_{\mathcal{D}(Q,W)}^1(x_{m-1}, x_m)[1] \otimes \dots \otimes \text{Hom}_{\mathcal{D}(Q,W)}^1(x_0, x_1)[1] \rightarrow \text{Hom}_{\mathcal{D}(Q,W)}^2(x_0, x_m)[1]$$

which are given by W_{m+1} , the $(m+1)$ th homogeneous part of W , via the natural pairing

$$(e_{m-1} \cdot S \cdot e_m)^* \otimes \dots \otimes (e_0 \cdot S \cdot e_1)^* \otimes e_0 \cdot S \cdot e_1 \otimes \dots \otimes e_{m-1} \cdot S \cdot e_m \otimes e_m \cdot S \cdot e_0 \rightarrow e_m \cdot S \cdot e_0.$$

Note that this definition results in the identity $W = W_{\mathcal{D}(Q,W)}|_{\text{End}^1(\bigoplus_{i \in V(Q)} x_i)}$. This category has a natural inner product $\text{Hom}_{\mathcal{D}(Q,W)}(x_i, x_j)[1] \otimes \text{Hom}_{\mathcal{D}(Q,W)}(x_j, x_i)[1] \rightarrow \mathbb{C}[-1]$ satisfying the cyclicity condition.

Let us say a few words on the connection between $\Gamma(Q, W)$ and $\mathcal{D}(Q, W)$. Using the projection to the degree zero part of $\mathcal{D}(Q, W)$, we can make $R_i := e_i R \cong \mathbb{C}$ into a (trivial) right A_∞ -module, which we will denote $R_{i, \mathcal{D}(Q,W)}$, and we get an object in $D_\infty^b(\mathcal{D}(Q, W)) \subset D_\infty(\mathcal{D}(Q, W)) := \text{Fun}_\infty(\mathcal{D}(Q, W)^{\text{op}}, \text{Vect}_{\mathbb{C}}^{\mathbb{Z}})$, the dg category of right A_∞ -modules over $\mathcal{D}(Q, W)$ with finite dimensional bounded cohomology. With the help of the bar construction one can show that there are quasi-isomorphisms $\text{RHom}_\infty(R_{i, \mathcal{D}(Q,W)}, R_{j, \mathcal{D}(Q,W)}) \simeq \text{Hom}_{\Gamma(Q,W)}(x_j, x_i) \simeq \text{RHom}_{D(\text{Mod-}\Gamma(Q,W))}(x_i, x_j)$, where we used the Yoneda embedding of $\Gamma(Q, W)$ into $D(\text{Mod-}\Gamma(Q, W))$ - in fact if one uses the bar construction to demonstrate the first of these quasi-isomorphisms, it is an equality. This establishes that $\Gamma(Q, W)$ and $\mathcal{D}(Q, W)$ are Koszul dual, and so we get similar quasi-isomorphisms after swapping them - i.e. there is a quasi-isomorphism

$\text{RHom}_{D^b(\text{Mod-}\Gamma(Q,W))}(R_{i, \Gamma(Q,W)}, R_{j, \Gamma(Q,W)}) \simeq \text{Hom}_{\mathcal{D}(Q,W)}(x_i, x_j)$. Hence we get a well defined functor¹⁰ $\text{RHom}(R_{\Gamma(Q,W)}, -) : D_{\text{cf}}(\Gamma(Q, W)) \rightarrow D_\infty(\mathcal{D}(Q, W))$ taking $R_{i, \Gamma(Q,W)}$ to a module quasi-isomorphic to x_i , considered as a $\mathcal{D}(Q, W)$ module, and a commutative diagram of dg categories

$$\begin{array}{ccc} D_{\text{cf}}(\text{Mod-}\Gamma(Q, W)) & \xrightarrow{\text{RHom}_\infty(R_{\Gamma(Q,W)}, -)} & D_\infty(\mathcal{D}(Q, W)) \\ \uparrow & & \uparrow \\ D_{\text{cf}}(\langle R_{i, \Gamma(Q,W)}, i \in V(Q) \rangle_{\text{thick}}) & \xrightarrow{\sim} & D_\infty(\langle x_i \in \text{Mod-}\mathcal{D}(Q, W), i \in V(Q) \rangle_{\text{thick}}) \\ \uparrow \sim & & \uparrow \sim \\ D_{\text{cf}}(\langle R_{i, \Gamma(Q,W)}, i \in V(Q) \rangle_{\text{triang}}) & \xrightarrow{\sim} & D_\infty(\langle x_i \in \text{Mod-}\mathcal{D}(Q, W), i \in V(Q) \rangle_{\text{triang}}) \end{array}$$

where for $S \subset \text{ob}(\text{Mod-}\Gamma(Q, W))$, $D_{\text{cf}}(\langle S \rangle_{\text{triang}})$ refers to the full subcategory of fibrant cofibrant dg $\Gamma(Q, W)$ modules M that are quasi-isomorphic to objects in the closure of S under taking triangles and shifts, and $D_{\text{cf}}(\langle S \rangle_{\text{thick}})$ is defined in the same way, except we take the closure under the operation of taking homotopy retracts too. Here, the leftmost vertical quasi-equivalence is due to the fact that both categories can be seen to be the full subcategory of the dg category of dg $\Gamma(Q, W)$ modules consisting of fibrant cofibrant objects with finite dimensional nilpotent homology. In particular, $D(\langle R_{i, \Gamma(Q,W)}, i \in V(Q) \rangle_{\text{triang}})$ is already closed under taking retracts. Passing to the homotopy category and restricting to degree zero modules, this amounts to the equivalence of abelian categories

$$\text{Mod}_{\text{nilp-}} A_{Q,W} \xrightarrow{\sim} H^0(D_{\text{cf}}(\langle R_{i, \Gamma(Q,W)}, i \in V(Q) \rangle_{\text{thick}}),$$

which provides the link between all this category theory and the geometry of the rest of the paper. Moreover, the four lower categories are quasi-equivalent to 3-Calabi-Yau categories - we will make this structure very explicit by considering the nice model, $\text{tw}(\mathcal{D}(Q, W))$, as described in [26], for the closure of $\{x_i \in \text{Mod-}\mathcal{D}(Q, W)\}$ under taking shifts and triangles. By a model of a cyclic A_∞ -category we mean a quasi-equivalent A_∞ -category. The reason to take care over the model we take is that there are many natural models for the triangulated closure of $\{x_i \in \text{Mod-}\mathcal{D}(Q, W)\}$ that do not have finite-dimensional morphism spaces (for example, compute on the left hand side of the above quasi-equivalences using the bar resolution) and so have no chance of having a bracket inducing a perfect pairing.

¹⁰We define this via a fibrant replacement of $R_{\Gamma(Q,W)}$.

Objects of $\mathbf{tw}(\mathcal{D}(Q, W))$ are given by pairs (T, α) , where $T = \bigoplus_{i=1}^n x_{a_i}[b_i] \in D_\infty(\mathcal{D}(Q, W)^{\text{op}})$ is a finite direct sum of right $\mathcal{D}(Q, W)$ -modules given by arbitrary shifts of objects $x_{a_i} \in \mathbf{ob}(\mathcal{D}(Q, W))$ covariantly embedded via the Yoneda embedding, and α is an element of $\bigoplus_{i < j} \text{Hom}_{\mathcal{D}(Q, W)}^{b_j - b_i + 1}(x_{a_i}, x_{a_j}) \simeq \bigoplus_{i < j} \text{RHom}_\infty^1(x_{a_i}[b_i], x_{a_j}[b_j]) \subset \text{RHom}_\infty^1(T, T)$ satisfying the Maurer–Cartan equation

$$\sum_{n \geq 1} b_{\mathcal{D}(Q, W), n}(\alpha, \dots, \alpha) = 0.$$

Given two pairs (T_1, α_1) and (T_2, α_2) , where $T_1 = \bigoplus_{i=1}^n x_{1, a_i}[b_{1, i}]$ and $T_2 = \bigoplus_{i=1}^m x_{2, a_i}[b_{2, i}]$, the graded vector space of homomorphisms $\text{Hom}_{\mathbf{tw}(\mathcal{D}(Q, W))}((T_1, \alpha_1), (T_2, \alpha_2))$ is given by $\bigoplus_{i, j} \text{Hom}_{\mathcal{D}(Q, W)}(x_{1, a_i}, x_{2, a_j})[b_{2, j} - b_{1, i}] \simeq \text{RHom}_\infty^0(T_1, T_2)$. Multiplication is twisted by setting

$$b_{\mathbf{tw}(\mathcal{D}(Q, W))}(f_n, \dots, f_1) = \sum b_{\mathcal{D}(Q, W)}(\alpha_n, \dots, \alpha_n, f_n, \alpha_{n-1}, \dots, \alpha_1, f_1, \alpha_0, \dots, \alpha_0)$$

where $f_i \in \text{Hom}_{\mathbf{tw}(\mathcal{D}(Q, W))}((T_{i-1}, \alpha_{i-1}), (T_i, \alpha_i))$. One may check that this is again a 3-Calabi–Yau category. For $f \in \text{Hom}_{\mathbf{tw}(\mathcal{D}(Q, W))}((\bigoplus_i x_{a_i}[b_i], \alpha_0), (\bigoplus_j x_{a_j}[b_j], \alpha_1))$ and $g \in \text{Hom}_{\mathbf{tw}(\mathcal{D}(Q, W))}((\bigoplus_j x_{a_j}[b_j], \alpha_1), (\bigoplus_i x_{a_i}[b_i], \alpha_0))$ one sets

$$\langle f, g \rangle := \sum_{i \in I, j \in J} \langle f_{ij}, g_{ji} \rangle,$$

where we denote by f_{ij} the degree $(b_j - b_i)$ morphism $x_{a_i} \rightarrow x_{a_j}$ induced by f .

In fact we will only be interested in the category $\mathbf{tw}_0(\mathcal{D}(Q, W))$, which is the full subcategory of $\mathbf{tw}(\mathcal{D}(Q, W))$ with objects given by pairs (T, α) , with T isomorphic to a direct sum of *unshifted* copies of the right modules $x_i \in \mathbf{ob}(\mathcal{D}(Q, W))$. Using $\text{RHom}_\infty(R_{\Gamma(Q, W)}, -)$ this in turn is an enrichment of the Abelian category of nilpotent modules over the Jacobi algebra $A_{Q, W}$.

Let $\mathfrak{TW}_{\mathbf{n}}$ be the moduli functor on finite type schemes defined as follows

$$\mathfrak{TW}_{\mathbf{n}}(X) := \{\text{pairs of rank } \sum_{i \in V(Q)} \mathbf{n}(i) \text{ vector bundles } \bigoplus_{i \in V(Q)} T_{\mathbf{n}(i)} \text{ on } X \text{ and}$$

$$\alpha \in \bigoplus_{i, j \in V(Q)} \text{Hom}_{\mathcal{D}(Q, W)}^1(x_i, x_j) \otimes T_{\mathbf{n}(i)}^* \otimes T_{\mathbf{n}(j)}$$

$$\text{such that } \sum_{n \geq 1} b_{\mathcal{D}(Q, W)}(\alpha, \dots, \alpha) = 0\}.$$

This moduli functor takes schemes over \mathbb{C} to sets of families of objects in $\mathbf{tw}_0(\mathcal{D}(Q, W))$. This is naturally made into a groupoid valued moduli functor, where the morphisms are defined via the conjugation action of $\prod_{i \in V(Q)} \text{GL}_{\mathbb{C}}(\mathbf{n}(i))$. There is a natural isomorphism of moduli functors $\mathfrak{TW}_{\mathbf{n}} \rightarrow \mathbf{nilp}_{\mathbf{n}}$, where

$$\begin{aligned} \mathbf{nilp}_{\mathbf{n}}(X) := & \{\text{vector bundles } T \text{ on } X \text{ with a } \mathcal{O}_X \otimes \Gamma(Q, W)\text{-action, nilpotent} \\ & \text{with respect to the } \Gamma(Q, W) \text{ part, such that for all } i \in V(Q), \\ & T \cdot e_i \text{ is a rank } \mathbf{n}(i) \text{ vector bundle.}\} \end{aligned}$$

The moduli functor $\mathbf{nilp}_{\mathbf{n}}$ is again a groupoid valued functor with morphisms given by conjugation, and it is represented by the stack $\mathcal{X}_{Q, W, \mathbf{n}}^{\text{nilp}}$.

4.2. Orientation data. There is one extra piece of data, aside from the 3-Calabi–Yau category $\mathbf{tw}_0(\mathcal{D}(Q_{-2}, W_d))$, that we need before we can apply the machinery of [29] to define and compute motivic Donaldson–Thomas invariants of (-2) -curves, which is the data (\mathcal{L}, ϕ) of an ind-constructible super (i.e. \mathbb{Z}_2 -graded) line bundle \mathcal{L} on $\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}}$ along with a chosen trivialisation of the tensor square $\phi : \mathcal{L}^{\otimes 2} \cong \mathbf{1}_{\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}}}$. Note that every constructible super line bundle has trivial tensor square, and so all the data here is in this choice of trivialisation (and the parity of the super line bundle). Such data is required to satisfy a cocycle condition (see Section 5.2 of [29]), ensuring that the integration map defined with respect to it (see equation (11)) is a $K(\text{St}^{\text{aff}} / \text{Spec}(\mathbb{C}))$ -algebra homomorphism, and is called orientation data in [29]. An isomorphism of orientation data is just an isomorphism of the underlying super line

bundles commuting with the trivialisations of the squares. Isomorphic choices will give rise to the same integration map (11). In fact for Q, W a finite quiver with arbitrary potential, $\mathcal{X}_{Q,W}^{\text{nilp}}$ comes with a natural choice of orientation data, which we briefly describe. Given an object $\eta = (T, \alpha)$ of $\mathbf{tw}_0(\mathcal{D}(Q, W))$, there is an explicit model of the cyclic A_∞ -algebra $\text{End}_{\mathbf{tw}(\mathcal{D}(Q, W))}(\eta)$, coming from the definition of $\mathbf{tw}(\mathcal{D}(Q, W))$. In particular, there is a differential¹¹ $b_{\alpha,1}$ on $\text{End}_{\mathbf{tw}(\mathcal{D}(Q, W))}^\bullet(\eta)$, and a nondegenerate inner product on $\text{End}_{\mathbf{tw}(\mathcal{D}(Q, W))}^1(T)/\text{Ker}(b_{\alpha,1})$ given by $\langle b_{\alpha,1}(\bullet), \bullet \rangle$. Across the family of possible α in the pair (T, α) , given by solutions to the Maurer–Cartan equation, we obtain a constructible vector bundle

$$(10) \quad \text{End}_{\mathbf{tw}(\mathcal{D}(Q, W))}^1(T)/\text{Ker}(b_{\mathbf{tw}(\mathcal{D}(Q, W)),1})$$

with nondegenerate quadratic form which we will denote by \overline{Q} . It is only a constructible vector bundle since the dimension of (10) jumps, due to the dependancy of $b_{\mathbf{tw}(\mathcal{D}(Q, W)),1}$ on α . Given a constructible super vector bundle \mathcal{V} on a stack \mathfrak{M} , one defines the superdeterminant $\mathbf{sDet}(\mathcal{V}) := \prod^{\dim(\mathcal{V})} (\bigwedge^{\text{top}} \mathcal{V}_{\text{even}} \otimes \bigwedge^{\text{top}} \mathcal{V}_{\text{odd}}^*)$, where here \prod denotes the change of parity functor. Say now \mathcal{V} has nondegenerate quadratic form $\overline{Q}_{\mathcal{V}}$, then we obtain a trivialisation of $\mathbf{sDet}(\mathcal{V})^{\otimes 2}$ since $\overline{Q}_{\mathcal{V}}$ establishes an isomorphism $\mathbf{sDet}(\mathcal{V}) \cong \mathbf{sDet}(\mathcal{V})^*$. In the present situation, orientation data on $\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}}$, considered as the moduli space of objects in $\mathbf{tw}_0(\mathcal{D}(Q_{-2}, W_d))$, is provided by the superdeterminant of (10), with the trivialisation of the tensor square provided by the nondegenerate inner product $\langle b_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d)),1}(\bullet), \bullet \rangle$. We will denote this choice of orientation data by τ_{Q_{-2}, W_d} .

Remark 4.1 ([6]). There are in general several choices for the orientation data of a 3-Calabi–Yau category. However in the case of the category $\mathbf{tw}_0(\mathcal{D}(Q, W))$ this range of choices is quite small, due to the constraint that the orientation data must satisfy the cocycle condition from [29]. In fact the orientation data is determined up to isomorphism entirely by its restriction to the simple modules x_i , for $i \in V(Q)$, and so one deduces that there are $2^{V(Q)}$ isomorphism classes of choices, giving rise to $2^{V(Q)}$ distinct integration maps, defined as in Theorem 4.7

Definition 4.2. A constructible 3-Calabi–Yau vector bundle on a scheme X is a constructible graded vector bundle V , along with degree one morphisms $b_n(V[1])^{\otimes n} \rightarrow V[1]$ and a degree minus one morphism $\langle \bullet, \bullet \rangle : V[1] \otimes V[1] \rightarrow \mathbf{1}_X$ satisfying the same conditions as a 3-Calabi–Yau category.

We recall the definition of a morphism of cyclic A_∞ -objects in the case of a constructible 3-Calabi–Yau vector bundle.

Definition 4.3. A morphism $f : V \rightarrow V'$ of constructible 3-Calabi–Yau vector bundles is a countable collection of morphisms $f_n : V[1]^{\otimes n} \rightarrow V'[1]$ satisfying the usual conditions of an A_∞ -morphism, as well as the extra conditions that $\langle \bullet, \bullet \rangle_{V'} \circ f_1 \otimes f_1 = \langle \bullet, \bullet \rangle_V$ and $\sum_{a+b=n} \langle \bullet, \bullet \rangle_{V'} \circ f_a \otimes f_b = 0$ for all $n \geq 3$.

To complete the definition of the integration map of [29] we need the following proposition.

Proposition 4.4 ([23]). There is a locally constructible formal isomorphism of A_∞ -vector bundles

$$(\text{End}_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))}^\bullet, b_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))}) \cong (\text{Ext}_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))}^\bullet \oplus V^\bullet, b')$$

on the stack $\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}}$ such that b'_i is dependent only on V^\bullet if $i = 1$, and b'_i is dependent only on $\text{Ext}_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))}^\bullet$ otherwise, and (V^\bullet, b'_1) is an acyclic complex. This splitting is unique up to isomorphisms of cyclic A_∞ -vector bundles.

Note that even though one starts with the data of a cyclic A_∞ -vector bundle, the splitting will only take place in the category of locally constructible cyclic A_∞ -vector bundles, since the dimension of the kernel of b_1 will jump in families. The reference [23] demonstrates this splitting in the case of cyclic A_∞ categories. Given a constructible 3-Calabi–Yau vector bundle V , we define the function W_{\min} as follows. Firstly, let E be a cyclic minimal model for V . Next, consider the Artin stack E^1/E^0 over X , given over a point $x \in X$ by taking the stack theoretic quotient of the trivial action of E_x^0 on E_x^1 (this is just what is known as a cone stack, see e.g. [2], where they arise in a similar context). We define W_{\min} to be the

¹¹Recall that $\text{End}_{\mathbf{tw}(\mathcal{D}(Q, W))}^\bullet(\eta) \cong \text{End}_{\mathcal{D}(Q, W)}^\bullet(T)$ as graded vector spaces, and does not depend on α , so in fact we obtain a family of differentials as we vary α .

function on this stack defined by W_E , the potential for the minimal part E . This potential is strictly speaking only defined up to a formal automorphism, which will not matter when it comes to considering motivic vanishing cycles.

Proposition 4.5. *Let V be a vector bundle on a scheme X , and let f be a polynomial function on V with trivial constant coefficient (i.e. f vanishes on X , considered as the zero section of V). Let g be another polynomial function on V vanishing on X , such that there exists a formal change of coordinates q on the vector bundle V around X such that $f = g \circ q$, considered as functions on a formal neighbourhood of X . Then $\phi_f^{\text{rel}}|_X = \phi_g^{\text{rel}}|_X$.*

Note that since we are dealing here with functions defined on vector bundles, we use the relative version of motivic vanishing cycles mentioned at the end of Subsection 3.3.

Proof. The proposition follows straight from the definition, since q induces \mathbb{G}_m -equivariant isomorphisms on arc spaces making the following diagram commute

$$\begin{array}{ccc} \mathcal{L}_n(V)|_X & \xrightarrow{q_*} & \mathcal{L}_n(V)|_X \\ & \searrow \pi \circ f_* & \swarrow \pi \circ g_* \\ & \mathbb{A}_{\mathbb{C}}^1 & \end{array}$$

where π is as in Subsection 3.3, and $\mathcal{L}_n(V)$ is as at the end of the same section. \square

Proposition 4.6. *Let A be a constructible 3-Calabi–Yau vector bundle on a scheme X , and let $f : E \rightarrow A$ be a quasi-isomorphism of constructible 3-Calabi–Yau vector bundles, with E minimal. Furthermore, assume that both of these 3-Calabi–Yau bundles have polynomial potential, i.e. their multiplications $b_{A,i}$ and $b_{E,i}$ vanish for sufficiently large i . Let $A = E \oplus V$ be a splitting of differential graded constructible vector bundles extending the inclusion f , and let $Q(x) = \langle x, b_{V,1}(x) \rangle$ be the induced quadratic function on V . Then there is an equality*

$$\phi_{W_E \oplus Q}^{\text{rel}}|_X = \phi_{W_A}^{\text{rel}}|_X$$

in $K^{\hat{\mu}}(\text{Var}/X)$.

Proof. We may choose a splitting $A = A_{\min} \oplus A_{\text{cont}}$ of A as in the cyclic minimal model theorem, with the underlying graded vector space of A_{\min} equal to $\text{Image}(f_1)$, and the underlying graded vector space of A_{cont} orthogonal to the underlying graded vector space of A_{\min} under the bracket $\langle \bullet, \bullet \rangle$. Then one may check that $\pi_{\min} \circ f$ is a cyclic A_{∞} isomorphism, where π_{\min} is the projection $A \rightarrow A_{\min}$. Extending this morphism along the inclusion $A_{\text{cont}} \rightarrow A$ gives a cyclic isomorphism $E \oplus A_{\text{cont}} \rightarrow A$. One may check that as a constructible vector bundle with quadratic form, A_{cont}^1 is isomorphic to the direct sum of $A^1/(\text{Ker}(b_1 : A^1 \rightarrow A^2))$, with quadratic form $\langle b_1(\bullet), \bullet \rangle$, and the vector space $\text{Image}(b_1 : A^0 \rightarrow A^1)$ with the zero function. The proposition then follows from Proposition 4.5. \square

Let Q, W be a quiver with polynomial potential. Given an element $[X \xrightarrow{f} \mathcal{X}_{Q,W,\mathbf{n}}^{\text{nilp}}] \in K(\text{Var}/\mathcal{X}_{Q,W,\mathbf{n}}^{\text{nilp}})$ Kontsevich and Soibelman define

$$(11) \quad \Phi_{Q,W}([X \xrightarrow{f} \mathcal{X}_{Q,W,\mathbf{n}}^{\text{nilp}}]) = \left(\int_X f^* \phi_{W_{\min} \oplus \overline{Q}_{\tau_{Q,W}}}^{\text{rel}} \right) \hat{\mathbf{e}}_{\mathbf{n}} \in K^{\hat{\mu}}(\text{St}^{\text{aff}}/\text{Spec}(\mathbb{C}))[[\mathbb{L}^{1/2}][[\hat{\mathbf{e}}_{\mathbf{m}}, \mathbf{m} \in \mathbb{N}^{V(Q)}]]$$

where $\overline{Q}_{\tau_{Q,W}}$ is a function $\overline{Q}_{\tau_{Q,W}}(z) = \overline{Q}_{\tau_{Q,W}}(z, z)$ on V , for some pair of ind-constructible vector bundle V on $\mathcal{X}_{Q,W,\mathbf{n}}^{\text{nilp}}$ and nondegenerate inner product $\overline{Q}_{\tau_{Q,W}}$ on V giving rise to the natural orientation data on $\mathcal{X}_{Q,W}^{\text{nilp}}$ arising from its realisation as the moduli space of objects in $\text{tw}_0(\mathcal{D}(Q, W))$. That is, under the natural identification $\mathbf{sDet}(V) \cong \mathbf{sDet}(V)^*$ induced by $\overline{Q}_{\tau_{Q,W}}$, we obtain the natural orientation data $\tau_{Q,W}$ on $\mathcal{X}_{Q,W,\mathbf{n}}^{\text{nilp}}$ given by (10) with its natural nondegenerate product. The function, W_{\min} is as defined after Proposition 4.4. The target is just the ring of formal power series in variables $\hat{\mathbf{e}}_{\mathbf{m}}$,

with the usual¹² multiplication $\hat{e}_{\mathbf{m}'} \cdot \hat{e}_{\mathbf{m}} = \hat{e}_{\mathbf{m}'+\mathbf{m}}$. One extends to a map $\Phi_{Q,W} : K(\text{St}^{\text{aff}}/\mathcal{X}_{Q,W}^{\text{nilp}}) \rightarrow K^{\hat{\mu}}(\text{St}^{\text{aff}}/\text{Spec}(\mathbb{C}))[[\mathbb{L}^{1/2}][[\hat{e}_{\mathbf{m}}, \mathbf{m} \in \mathbb{N}^{V(Q)}]]$ by $K(\text{St}^{\text{aff}}/\text{Spec}(\mathbb{C}))$ -linearity and Proposition 3.1.

For general 3-Calabi–Yau categories the following is only a theorem if one is able to work with motivic vanishing cycles of formal functions and prove the motivic integral identity of [29]. For the former, see the comment immediately following the theorem. For the latter, see [?] or [41].

Theorem 4.7 ([29]). *The morphism $\Phi_{Q,W} : K(\text{St}^{\text{aff}}/\mathcal{X}_{Q,W}^{\text{nilp}}) \rightarrow K^{\hat{\mu}}(\text{St}^{\text{aff}}/\text{Spec}(\mathbb{C}))[[\mathbb{L}^{1/2}][[\hat{e}_{\mathbf{n}}|\mathbf{n} \in \mathbb{N}^{V(Q)}]]$ is a $K(\text{St}^{\text{aff}}/\text{Spec}(\mathbb{C}))$ -algebra homomorphism.*

The issue with formal functions is not a serious one in our case. Let $X/\text{Spec}(\mathbb{C})$ be a finite type scheme. There is a multiplication \star_X on $K^{\mathbb{G}_m,n}(\text{St}^{\text{aff}}/\mathbb{A}_X^1)$ given by $[Y_1 \xrightarrow{f_1} \mathbb{A}_\mathbb{C}^1 \times X] \star_X [Y_2 \xrightarrow{f_2} \mathbb{A}_\mathbb{C}^1 \times X] = [Y_1 \times_X Y_2 \xrightarrow{p} \mathbb{A}_\mathbb{C}^1 \times X]$, where the fibre product is with respect to the morphisms $\pi_X \circ f_1$ and $\pi_X \circ f_2$, and the map p is defined by $(y_1, y_2) \mapsto (\pi_{\mathbb{A}_\mathbb{C}^1} \circ f_1(y_1) + \pi_{\mathbb{A}_\mathbb{C}^1} \circ f_2(y_2), \pi_X \circ f_1(y_1))$. This multiplication descends to $K^{\hat{\mu}}(\text{St}^{\text{aff}}/X)$. We make the following definition.

Definition 4.8. *Let V be a vector bundle on the scheme X , and let f be a formal function on V . Furthermore, assume that there exists a vector bundle V' on X and a quadratic form \bar{Q} on V' , such that there is a formal change of coordinates on $V \oplus V'$ taking $f \boxplus \bar{Q}$ to a polynomial function g (here we abuse notation and write $\bar{Q}(z) = \bar{Q}(z, z)$). Then we define $\phi_f^{\text{rel}}|_X = \phi_g^{\text{rel}}|_X \star_X \phi_{\bar{Q}}^{\text{rel}}|_X$.*

Note that in the above definition one may show, using the motivic Thom-Sebastiani theorem [8], that $\phi_Q^{\text{rel}} \star_X \phi_Q^{\text{rel}} = [X \rightarrow X]$, the unit of the ring $(K^{\hat{\mu}}(X), \star_X)$, and so we may make the more natural looking definition $\phi_f^{\text{rel}}|_X = \phi_g^{\text{rel}}|_X \star_X (\phi_{\bar{Q}}^{\text{rel}}|_X)^{-1}$ instead. If one works with the minimal potentials of objects in the category of modules over a Ginzburg differential graded algebra for a quiver with polynomial potential, one only needs to deal with formal functions f satisfying the conditions of Definition 4.8.

There is a more down to earth way to define the integration map for the Hall algebra $K(\text{St}^{\text{aff}}/\mathcal{X}_{Q,W}^{\text{nilp}})$. In fact this second way extends without any effort to an integration map

$$\Phi_{\text{BBS},Q,W} : K(\text{St}^{\text{aff}}/\mathcal{X}_{Q,W}) \rightarrow K^{\hat{\mu}}(\text{St}^{\text{aff}}/\text{Spec}(\mathbb{C}))[[\hat{e}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^{V(Q)}]]$$

exploited by Behrend, Bryan and Szendrői to define and calculate motivic Donaldson–Thomas counts for Hilbert schemes of points on \mathbb{C}^3 in [1]. The Hodge theoretic version of this construction is a part of [30], see also [9]. One defines, similarly to the Kontsevich–Soibelman integration map,

$$\Phi_{\text{BBS},Q,W} : [X \xrightarrow{f} \mathcal{X}_{Q,W,\mathbf{n}}] \mapsto \int_X f^* \phi_{\text{tr}(W)} \hat{e}_{\mathbf{n}}.$$

The following comparison theorem will be used in the proof of Propositions 5.2 and 5.3.

Proposition 4.9 ([6]). *There is an equality of maps $\Phi_{\text{BBS},Q,W}|_{K(\text{St}^{\text{aff}}/\mathcal{X}_{Q,W}^{\text{nilp}})} = \Phi_{Q,W}$.*

5. MOTIVIC DONALDSON–THOMAS INVARIANTS OF (-2) -CURVES

5.1. The calculation of the invariants. We are finally able to calculate the motivic Donaldson–Thomas invariants of the category of nilpotent modules over A_{Q_{-2},W_d} . Firstly, we pick a stability condition $\gamma : \mathbb{N}^{V(Q_{-2})} \rightarrow \mathbb{H}_+$, where \mathbb{H}_+ is the set $\{r \cdot e^{i\theta} | r \in \mathbb{R}_{>0}, \theta \in (0, \pi]\}$. We make the genericity assumption that γ does not map the whole of $\mathbb{N}^{V(Q_{-2})}$ onto the same ray in \mathbb{C} . As a result of the fact that there is a symmetry of the pair (Q_{-2}, W_d) that interchanges the vertices, it will not end up mattering which stability condition γ we pick. We recall, regardless, that a module M of slope $\zeta := \arg(\gamma(\dim(M)))$ is called (semi)stable if for all proper submodules $N \subset M$, $\arg(\gamma(\dim(N))) (\leq) \arg(\gamma(\dim(M)))$, where the bracket denotes the fact that for semistability we only require the weak inequality.

Lemma 5.1. *Let M be a semistable nilpotent A_{Q_{-2},W_d} -module with slope ζ . Then M is given by repeated extension by stable modules M_α of slope ζ , such that $M_\alpha \cdot (X + Y) = 0$.*

¹²In fact usually one would twist the multiplication by some power of $-\mathbb{L}^{1/2}$, but we escape this necessity as we only work with symmetric quivers.

and in the case $d \geq 2$, W_{\min} is given by x^{d+1} , since the universal deformation of \mathcal{O}_C is over the Artinian ring $\mathbb{C}[x]/(x^d)$. The 3-Calabi–Yau category of semistable A_{Q_{-2}, W_d} -modules of slope ζ is, then, quasi-isomorphic as a cyclic category to the category $\mathbf{tw}_0(Q_L^1, X^{d+1})$, where for $a \in \mathbb{N}$, Q_L^a is the quiver with one vertex and a loops.

We claim that the orientation data τ_{Q_{-2}, W_d} over M , considered as a point in $\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}}$, is trivial if and only if $d \geq 2$. For this, note that there are precisely two isomorphism classes of orientation data over a point; given two super line bundles \mathcal{V}_1 and \mathcal{V}_2 over a point, i.e. vector spaces with parity, and isomorphisms $\mathcal{V}_i^{\otimes 2} \xrightarrow{\eta_i} \mathbb{C}$, there is an isomorphism $\mathcal{V}_1 \xrightarrow{f} \mathcal{V}_2$ such that $\eta_2 \circ (f \otimes f) = \eta_1$ if and only if the parity of \mathcal{V}_1 is the same as that of \mathcal{V}_2 . It is sufficient, then, to show that $\dim(\text{End}_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))}^1(M) / \text{Ker}(b_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d)), 1}))$ is even if and only if $d \geq 2$. This follows from the following equations:

$$(14) \quad \dim(\text{Hom}_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))}^0(M, M)) = \mathbf{n}(0)^2 + \mathbf{n}(1)^2 = 1 \quad (\text{modulo } 2)$$

$$(15) \quad \dim(\text{Hom}_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))}^1(M, M)) = 4\mathbf{n}(0)\mathbf{n}(1) + \mathbf{n}(0)^2 + \mathbf{n}(1)^2 = 1 \quad (\text{modulo } 2)$$

$$(16) \quad \dim(\text{Ext}_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))}^0(M, M)) = 1. \quad (\text{modulo } 2)$$

The first two identities follow from the definitions of homomorphism spaces in $\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))$, and the identity (16) follows from the fact that M is stable and hence simple. We deduce that

$$1 = \dim(\text{Ext}_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))}^1(M, M)) + \dim(\text{End}_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d))}^1(M) / \text{Ker}(b_{\mathbf{tw}(\mathcal{D}(Q_{-2}, W_d)), 1})) \quad (\text{modulo } 2).$$

Thus, the right hand summand has even dimension if and only if $d \geq 2$, and the claim is proved.

Now one can see directly that the orientation data assigned to the unique simple object s_0 of the category $\mathbf{tw}_0(\mathcal{D}(Q_L^1, X^{d+1}))$ is trivial if and only if $d \geq 2$, since $\text{Ker}(b_1 : \text{End}_{\mathbf{tw}(\mathcal{D}(Q_L^1))}^1(s_0) \rightarrow \text{End}_{\mathbf{tw}(\mathcal{D}(Q_L^1))}^2(s_0))$ is trivial if $d \geq 2$, otherwise this differential is an isomorphism. So as well as having a cyclic A_∞ -isomorphism Ξ from the subcategory of $\mathbf{tw}_0(Q_{-2}, W_d)$ generated by M under extensions to the category $\mathbf{tw}_0(Q_L^1, W^{d+1})$, we have an isomorphism of orientation data $\Xi^*(\tau_{Q_L^1, X^{d+1}}) \cong \tau_{Q_{-2}, W_d}$. It follows that

$$\begin{aligned} \Phi_{Q_{-2}, W_d}([\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}, \zeta-\text{ss}}]) &= \Phi_{Q_L^1, X^{d+1}}([\mathcal{X}_{Q_L^1, X^{d+1}}^{\text{nilp}}])_{\hat{e}_a \mapsto \hat{e}_{an}} \\ &= \Phi_{\text{BBS}, Q_L^1, X^{d+1}}([\mathcal{X}_{Q_L^1, X^{d+1}}^{\text{nilp}}])_{\hat{e}_a \mapsto \hat{e}_{an}} \\ &= \Phi_{\text{BBS}, Q_L^1, X^{d+1}}([\mathcal{X}_{Q_L^1, X^{d+1}}])_{\hat{e}_a \mapsto \hat{e}_{an}} \end{aligned}$$

where for the penultimate equation we used Proposition 4.9, and for the final equation we use the fact that all finite-dimensional representations of $H^0(\Gamma(Q_L^1, X^{d+1}))$ are nilpotent. The result now follows by the main result of [7]. \square

Proposition 5.3. *There is an equation of generating series*

$$\Phi_{Q_{-2}, W_d}([\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}, \zeta-\text{ss}}]) = \text{Sym} \left(\sum_{n \geq 1} \frac{\mathbb{L}^{-1/2} + \mathbb{L}^{-3/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \hat{e}_{(n, n)} \right)$$

for $\arg(\gamma((1, 1))) = \zeta$.

Proof. The simple stable nilpotent modules with dimension vector $(1, 1)$ are given by choosing two linear maps $\theta(A)$ and $\theta(B)$, from \mathbb{C} to \mathbb{C} , not both equal to zero. These modules correspond to the structure sheaves of points on the exceptional curve $C_d \subset Y_d$ under the derived equivalence (3). Let Y_n° be the subscheme of $\prod_{a \in E(Q_{-2})} \text{Hom}(\mathbb{C}^{\mathbf{n}(t(a))}, \mathbb{C}^{\mathbf{n}(s(a))})$, for $\mathbf{n} = (n, n)$, the points of which satisfy the condition that the linear map assigned to A is an isomorphism, and the Harder–Narasimhan filtration of the associated module only contains modules with dimension vector $(1, 1)$. The action of $\text{Gl}_{\mathbb{C}}(\mathbf{n}(1))$ on Y_n° is free. Taking the quotient by this action corresponds to forgetting the data of the isomorphism A , and identifying the two vertices of the quiver Q_{-2} , and so $Y_n^\circ / (\text{Gl}_{\mathbb{C}}(n) \times \text{Gl}_{\mathbb{C}}(n)) \cong \mathcal{Y}_{Q_5^L, n}$, where Q_5^L is

the five loop quiver, with loops labelled B, C, D, X, Y . Furthermore, under the open inclusion $\mathcal{Y}_{Q_5^L, n} \hookrightarrow \mathcal{Y}_{Q_{-2}, (n, n)}$, the function $\text{tr}(W_d)$ pulls back to the function $\text{tr}(W_d^\circ)$, where W_d° is the superpotential

$$W_d^\circ = \frac{1}{d+1}X^{d+1} - \frac{1}{d+1}Y^{d+1} - XC + XDB + YC - YBD.$$

If we define $\mathcal{X}_{Q_{-2}, W_d, \mathbf{n}}^{\text{nilp}, \zeta\text{-ss}, \circ}$ to be a substack of $\mathcal{X}_{Q_{-2}, W_d, \mathbf{n}}$, the points of which are ζ -semistable nilpotent A_{Q_{-2}, W_d} -modules M such that $\theta(A)$ is an isomorphism, then we have shown that $\mathcal{X}_{Q_{-2}, W_d, \mathbf{n}}^{\text{nilp}, \zeta\text{-ss}, \circ}$ is naturally a substack of $\mathcal{X}_{Q_5^L, W_d^\circ, n} \subset \mathcal{Y}_{Q_5^L, n}$, and we identify it as the stack of representations for the Jacobi algebra associated to (Q_5^L, W_d°) such that all loops apart from B act via nilpotent linear maps. Under the derived equivalence (3) this is the substack of coherent sheaves supported on the exceptional curve $C_d \subset Y_d$, away from a fixed point, which we will denote by p .

Denote by $\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}, \zeta\text{-ss}, p}$ the stack of modules for A_{Q_{-2}, W_d} which are supported at the point p under the derived equivalence (3). Then since every torsion sheaf on \mathbb{P}^1 splits uniquely as a direct sum of a coherent sheaf supported at p and a coherent sheaf supported away from p , there is an identity in the motivic Hall algebra $(\text{K}(\text{St}^{\text{aff}}/\mathcal{X}_{Q_{-2}, W_d}), \star)$

$$(17) \quad [\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}, \zeta\text{-ss}}] = [\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}, \zeta\text{-ss}, \circ}] \star [\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}, \zeta\text{-ss}, p}].$$

Now note that there is a splitting

$$(18) \quad W_d^\circ = XDB - XBD + (X - Y)(BD - C + \frac{1}{d+1}X^d + X^{d-1}Y + \dots + Y^d)$$

We deduce that after giving $Y_{Q_5^L, n}$ the coordinates $X, D, B, Y' = X - Y, C = BD - C + \frac{1}{d+1}X^d + X^{d-1}Y + \dots + Y^d$, we have

$$(19) \quad \Phi_{Q_{-2}, W_d}(\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}, \zeta\text{-ss}, \circ}) = \sum_{n \geq 0} \left(\int_{\mathcal{X}_{Q_{-2}, W_d, (n, n)}^{\text{nilp}, \zeta\text{-ss}, \circ} \subset \mathcal{Y}_{Q_{-2}, (n, n)}} \phi_{\text{tr}(W_d)} \right) \hat{e}_{(n, n)}$$

$$(20) \quad = \sum_{n \geq 0} [\text{Gl}_{\mathbb{C}}(n)]^{-1} \mathbb{L}^{n^2/2} \left(\int_{\mathcal{X}_{Q_{-2}, W_d, \mathbf{n}}^{\text{nilp}, \zeta\text{-ss}, \circ} \subset Y_{Q_5^L, n}} \phi_{\text{tr}(XDB - XBD) \oplus \text{tr}(Y'C')} \right) \hat{e}_{(n, n)}$$

$$(21) \quad = \sum_{n \geq 0} [\text{Gl}_{\mathbb{C}}(n)]^{-1} \mathbb{L}^{n^2/2} \left(\int_{\{X \text{ and } D \text{ nilpotent}\} \subset Y_{Q_5^L, n}} \phi_{\text{tr}(XDB - XBD)} \right) \hat{e}_{(n, n)}$$

Here (19) comes from the comparison theorem (Theorem 4.9), and (21) comes from applying the motivic Thom-Sebastiani theorem. Now, giving the coordinates X, D, B weights 0, 0 and 1 respectively, and applying the weight one version of Conjecture 5.5 (which is a theorem), with Z' the scheme of pairs of

matrices labelled by X and D , we obtain

$$(22) \quad \Phi_{Q_{-2}, W_d}(\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}, \zeta^{-\text{ss}, o}}) = \sum_{n \geq 0} [\text{Gl}_{\mathbb{C}}(n)]^{-1} \mathbb{L}^{-n^2} (\{X \text{ and } D \text{ nilpotent, } XD \neq DX\} (\mathbb{L}^{n^2-1} - \mathbb{L}^{n^2-1}) -$$

$$(23) \quad - \{X \text{ and } D \text{ nilpotent, } XD = DX\} \mathbb{L}^{n^2}) \hat{e}_{(n,n)}$$

$$(24) \quad = \sum_{n \geq 0} [\text{Gl}_{\mathbb{C}}(n)]^{-1} C_{n, \text{nilp}} \hat{e}_{(n,n)}$$

$$(25) \quad = \sum_{n \geq 0} C_{n, \text{nilp}} \hat{e}_{(n,n)}$$

$$(26) \quad = \left(\sum_{n \geq 0} C_n \hat{e}_{(n,n)} \right)^{-\mathbb{L}^2}$$

$$(27) \quad = \text{Sym} \left(\sum_{n \geq 1} \frac{\mathbb{L}^{-1/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \hat{e}_{(n,n)} \right)$$

where C_n is the variety of pairs of commuting $n \times n$ matrices, \mathcal{C}_n is its quotient under the conjugation action of $\text{Gl}_{\mathbb{C}}(n)$, which one can think of as the stack of length n coherent sheaves on \mathbb{C}^2 and $C_{n, \text{nilp}}$ and $\mathcal{C}_{n, \text{nilp}}$ are the variety and stack, respectively, of nilpotent commuting matrices, the second of which one should think of as the stack of coherent sheaves on \mathbb{C}^2 supported at the origin. Then (26) follows from the definition of the power structure in Section 3.1, and (27) follows from the results of [11] and [1]. Similarly one deduces that

$$\Phi_{Q_{-2}, W_d}(\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}, \zeta^{-\text{ss}, p}}) = \text{Sym} \left(\sum_{n \geq 1} \frac{\mathbb{L}^{-3/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \hat{e}_{(n,n)} \right)$$

and now the result follows from applying the integration map to the Hall algebra identity (17). \square

The following theorem then follows from applying the integration map to the Harder–Narasimhan identity (12) in $K(\text{St}^{\text{aff}} / \mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}})$.

Theorem 5.4. *For any stability function $\gamma : \mathbb{N}^{V(Q_{-2})} \rightarrow \mathbb{H}_+$ there is an equality in $K^{\hat{\mu}}(\text{St}^{\text{aff}} / \text{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}][[\hat{e}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^{V(Q_{-2})}]]$:*

$$\Phi_{Q_{-2}, W_d}(\mathcal{X}_{Q_{-2}, W_d}^{\text{nilp}}) = \text{Sym} \left(\sum_{n \geq 0} \frac{\mathbb{L}^{-1/2}(1 - [\mu_{d+1}])}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} (\hat{e}_{(n, n+1)} + \hat{e}_{(n+1, n)}) + \sum_{n \geq 1} \frac{\mathbb{L}^{-1/2} + \mathbb{L}^{-3/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \hat{e}_{(n, n)} \right).$$

In particular, the motivic Donaldson–Thomas invariants $\Omega_{\gamma}^{\text{nilp}}$ counting nilpotent A_{Q_{-2}, W_d} -modules (for any γ) are given by

$$\Omega_{\gamma}^{\text{nilp}}(\mathbf{n}) = \begin{cases} (1 - [\mu_{d+1}]) \cdot \mathbb{L}^{-1/2} & \text{if there exists } n \in \mathbb{N} \text{ such} \\ & \text{that } \mathbf{n} = (n, n+1) \text{ or } \mathbf{n} = (n+1, n), \\ \mathbb{P}^1 \cdot \mathbb{L}^{-3/2} & \text{if there exists } n \in \mathbb{N} \text{ such that } \mathbf{n} = (n, n). \end{cases}$$

5.2. Calculation using equivariant vanishing cycles. We repeat the above calculations, but this time the other side of the comparison theorem. It is more natural there to work out the Donaldson–Thomas invariants for the category of finite dimensional A_{Q_{-2}, W_d} -modules, not just the nilpotent ones. First we recall the following conjecture from [7].

Conjecture 5.5. *Let Z' be a smooth scheme with trivial \mathbb{G}_m -action, and let $\mathbb{A}_{\mathbb{C}}^n$ carry a \mathbb{G}_m -action with nonnegative weights. Let $Z = Z' \times \mathbb{A}_{\mathbb{C}}^n$ with the induced \mathbb{G}_m -action, and let $f : Z \rightarrow \mathbb{A}_{\mathbb{C}}^1$ be a \mathbb{G}_m -equivariant function, with \mathbb{G}_m acting on the target with weight $s > 0$. Then there is an equality in $K^{\hat{\mu}}(\text{St}^{\text{aff}} / Z')$*

$$(28) \quad q_* \phi_f = \mathbb{L}^{-\dim(Z)/2} ([f^{-1}(0)] - [f^{-1}(1)]),$$

where $f^{-1}(0)$ carries the trivial $\hat{\mu}$ -action, and the $\hat{\mu}$ -action on $f^{-1}(1)$ is given by the natural μ_s -action, and both are considered as varieties over Z' via the projection $q : Z \rightarrow Z'$. Equivalently, there is an identity in $\lim_{n \rightarrow \infty} K^{\mathbb{G}_m, n}(\text{St}^{\text{aff}} / \mathbb{A}_{Z'}^1)$

$$(29) \quad q_* \phi_f = \mathbb{L}^{-\dim(Z)/2} [Z \xrightarrow{f \times q} \mathbb{A}_{Z'}^1].$$

This conjecture follows from the proof of Theorem 5.9 in [7], under the assumption that the weights on $\mathbb{A}_{\mathbb{C}}^n$ are all less than or equal to one.

Theorem 5.6. *For $d \leq 2$, there is an identity*

$$\Phi_{\text{BBS}, Q_{-2}, W_d}([\mathcal{X}_{Q_{-2}, W_d}]) = \text{Sym} \left(\sum_{n \geq 0} \frac{\mathbb{L}^{-1/2}(1 - [\mu_{d+1}])}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} (\hat{e}_{(n, n+1)} + \hat{e}_{(n+1, n)}) + \sum_{n \geq 1} \frac{\mathbb{L}^{3/2} + \mathbb{L}^{1/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \hat{e}_{(n, n)} \right)$$

in $K^{\hat{\mu}}(\text{St}^{\text{aff}} / \text{Spec}(\mathbb{C}))[\mathbb{L}^{1/2}][[\hat{e}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^{V(Q-2)}]]$. Assuming Conjecture 5.5 this identity holds for all d . It follows that the motivic Donaldson–Thomas invariants $\Omega_{\gamma}(\mathbf{n})$ (which don't depend on γ) are given by¹⁴

$$\Omega_{\gamma}(\mathbf{n}) = \begin{cases} (1 - [\mu_{d+1}]) \cdot \mathbb{L}^{-1/2} & \text{if there exists } n \in \mathbb{N} \text{ such} \\ & \text{that } \mathbf{n} = (n, n+1) \text{ or } \mathbf{n} = (n+1, n), \\ [Y_d]_{\text{virt}} := \mathbb{L}^{\frac{-\dim(Y_d)}{2}} \cdot [Y_d] & \text{if there exists } n \in \mathbb{N} \text{ such that } \mathbf{n} = (n, n). \end{cases}$$

Proof. For $\beta = (\beta_n, \dots, \beta_1)$ a path in a quiver Q , and for

$$M \in Y_{Q, \mathbf{n}} := \prod_{a \in E(Q)} \text{Hom}(\mathbb{C}^{\mathbf{n}(t(a))}, \mathbb{C}^{\mathbf{n}(s(a))})$$

we write $M(\beta) = M(\beta_n) \circ \dots \circ M(\beta_1)$. We let \mathbb{G}_m act on $Y_{Q_{-2}, \mathbf{n}}$ via

$$(z \cdot M)(E) = z^{\iota(E)} \cdot M(E),$$

where $\iota(E) = 1$ if $E = X, Y, A, B$, and $\iota(E) = d - 1$ if $E = C, D$. Then $\text{tr}(W_d) : Y_{Q_{-2}, \mathbf{n}} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ is \mathbb{G}_m -equivariant, after giving $\mathbb{A}_{\mathbb{C}}^1$ the weight $(d + 1)$ -action. It follows from our assumption $d \leq 2$, or from the conjecture, for general d , that

$$\int_{\mathcal{X}_{Q_{-2}, W_d, \mathbf{n}}} \phi_{\text{tr}(W_d)} = \int_{Y_{Q_{-2}, \mathbf{n}}} \phi_{\text{tr}(W_d)} = [Y_{Q_{-2}, \mathbf{n}} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] \cdot \mathbb{L}^{-2\mathbf{n}(0)\mathbf{n}(1)} \cdot [\text{Gl}_{\mathbb{C}}(\mathbf{n}(0)) \times \text{Gl}_{\mathbb{C}}(\mathbf{n}(1))]^{-1}$$

in $K^{\mathbb{G}_m, d+1}(\text{St}^{\text{aff}} / \mathbb{A}_{\mathbb{C}}^1)$ (in fact all subsequent calculations will take place in this ring). The first of these equalities follows from the fact that the motive $\phi_{\text{tr}(W_d)}$ is supported on the critical locus of $\text{tr}(W_d)$, which is just $\mathcal{X}_{Q_{-2}, W_d, \mathbf{n}}$.

For a set of edges $E' \subset E(Q)$ let $Q \setminus E'$ be the quiver obtained by deleting the edges of E' (this quiver has the same vertex set as Q). If W is a potential on $\mathbb{C}Q$, we denote by $W \setminus E'$ the potential on $Q \setminus E'$ obtained by changing the coefficient of any term in W containing any edge of E' to zero. By abuse of notation we will often denote the potential $W \setminus E'$ on $Q \setminus E'$ by W . There is a natural projection $\pi_C : Y_{Q_{-2}, \mathbf{n}} \rightarrow Y_{Q_{-2} \setminus \{C\}, \mathbf{n}}$ given by forgetting the data $M(C)$. We consider this as the projection from the total space of a rank $\mathbf{n}(0) \cdot \mathbf{n}(1)$ vector bundle which we denote \tilde{C} . There is an obvious equality

$$[Y_{Q_{-2}, \mathbf{n}} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] = [\pi_C^{-1} Y_{Q_{-2} \setminus \{C\}, \mathbf{n}, M(AX)=M(YA)} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] + [\pi_C^{-1} Y_{Q_{-2} \setminus \{C\}, \mathbf{n}, M(AX) \neq M(YA)} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1].$$

The restriction of the vector bundle \tilde{C} to $Y_{Q_{-2} \setminus \{C\}, \mathbf{n}, M(AX) \neq M(YA)}$ has a rank $(\mathbf{n}(0) \cdot \mathbf{n}(1) - 1)$ sub-bundle \tilde{C}_0 , given by those choices of $M(C)$ such that $\text{tr}(M(CAX) - M(CYA)) = 0$. The action of \mathbb{G}_m on $Y_{Q_{-2} \setminus \{C\}, \mathbf{n}, M(AX) \neq M(YA)}$ is free, and from the corresponding non-equivariant statement on the quotient we deduce that after \mathbb{G}_m -equivariant constructible decomposition of the base $Y_{Q_{-2} \setminus \{C\}, \mathbf{n}, M(AX) \neq M(YA)}$, the inclusion $\tilde{C}_0 \subset \tilde{C}|_{Y_{Q_{-2} \setminus \{C\}, \mathbf{n}, M(AX) \neq M(YA)}}$ splits with trivial complement, and we may write

$$(30) \quad [\pi_C^{-1} Y_{Q_{-2} \setminus \{C\}, \mathbf{n}, M(AX) \neq M(YA)} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] = [\tilde{C}_0 \times \mathbb{A}_{\mathbb{C}}^1 \xrightarrow{\text{tr}(W_d \setminus C) + \pi_{\mathbb{A}_{\mathbb{C}}^1}} \mathbb{A}_{\mathbb{C}}^1]$$

¹⁴The explicit description given in Section 2 shows that Y_d is a locally trivial fibre bundle over the exceptional curve $C_d \cong \mathbb{P}_{\mathbb{C}}^1$ with fibre $\mathbb{A}_{\mathbb{C}}^2$, and so $[Y_d] = (\mathbb{L}^1 + 1)\mathbb{L}^2$ in $K^{\mu}(\text{St}^{\text{aff}} / \text{Spec}(\mathbb{C}))$. The transition functions are linear only for $d = 1$.

where we have abused notation by identifying \tilde{C}_0 with its constructible decomposition. After a change of coordinate we may write the right hand side of (30) as $[\tilde{C}_0 \times \mathbb{A}_{\mathbb{C}}^1 \xrightarrow{\pi_{\mathbb{A}_{\mathbb{C}}^1}} \mathbb{A}_{\mathbb{C}}^1]$. Clearly this belongs to \mathfrak{I}_{d+1} . We deduce that

$$\begin{aligned} [Y_{Q_{-2}, \mathbf{n}} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] &= [\pi_C^{-1} Y_{Q_{-2} \setminus \{C\}, \mathbf{n}, M(AX)=M(YA)} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] \\ &= \mathbb{L}^{\mathbf{n}(0) \cdot \mathbf{n}(1)} \cdot [Y_{Q_{-2} \setminus \{C\}, \mathbf{n}, M(AX)=M(YA)} \xrightarrow{\text{tr}(W_d \setminus C)} \mathbb{A}_{\mathbb{C}}^1] \end{aligned}$$

and similarly

$$[Y_{Q_{-2}, \mathbf{n}} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] = \mathbb{L}^{2\mathbf{n}(0) \cdot \mathbf{n}(1)} \cdot [E_{Q_{\text{Kron}}, \mathbf{n}} \xrightarrow{\text{tr}(X^{d+1} - Y^{d+1})} \mathbb{A}_{\mathbb{C}}^1].$$

where we define

$$E_{Q_{\text{Kron}}, \mathbf{n}} = Y_{Q_{-2} \setminus \{C, D\}, \mathbf{n}, \substack{M(AX)=M(YA) \\ M(BX)=M(YB)}}$$

and denote the quotient stack $\mathcal{E}_{Q_{\text{Kron}}, \mathbf{n}} := E_{Q_{\text{Kron}}, \mathbf{n}} / (\text{Gl}_{\mathbb{C}}(\mathbf{n}(0)) \times \text{Gl}_{\mathbb{C}}(\mathbf{n}(1)))$. It represents pairs (M, ξ) , where M is a right $A_{Q_{\text{Kron}}}$ -module with dimension vector \mathbf{n} , and $\xi = X + Y$ is an endomorphism of M , where Q_{Kron} is the Kronecker quiver with two vertices x_0 and x_1 , and two arrows A, B , both going from x_0 to x_1 .

Let $\gamma : \mathbb{N}^{V(Q_{\text{Kron}})} \rightarrow \mathbb{H}_+$ be such that $\arg(\gamma((1, 0))) < \arg(\gamma((0, 1)))$. Then the γ -stable modules for Q_{Kron} are as pictured in Figure 2. For each slope ζ there is at most one stable $A_{Q_{\text{Kron}}}$ -module, and for M_1, M_2 semistable $A_{Q_{\text{Kron}}}$ -modules with the slope $\arg(\gamma(\dim(M_2)))$ of M_2 smaller than that of M_1 , $\text{Ext}_{A_{Q_{\text{Kron}}}}^1(M_2, M_1) = 0$. It follows that a Harder–Narasimhan type is fixed by a series of vectors $\alpha = (\alpha_1, \dots, \alpha_t)$ such that $\gamma(\alpha_i)$ is descending, and each α_i is some multiple of the vectors $(n, n \pm 1)$ and $(1, 1)$, and that a module is determined up to isomorphism by its Harder–Narasimhan type, and is a direct sum of the modules appearing as quotients of successive terms in the Harder–Narasimhan filtration. Let \mathfrak{N}_{γ} be the set of all possible Harder–Narasimhan types. It follows that $\mathcal{E}_{Q_{\text{Kron}}, \alpha}$, the stack of pairs (M, ξ) such that M has Harder–Narasimhan type α , is given by a stack theoretic quotient $\text{Hom}(M, M) / \text{Aut}(M, M)$, for M a fixed module of Harder–Narasimhan type α . We may write

$$\text{Hom}(M, M) \cong \prod_{t \geq i \geq j \geq 1} \text{Hom}(M_{\alpha_i}, M_{\alpha_j}),$$

where for each i , M_{α_i} is the unique semistable module of dimension vector α_i , and

$$\text{Aut}(M, M) \cong \prod_{t \geq i > j \geq 1} \text{Hom}(M_{\alpha_i}, M_{\alpha_j}) \times \prod_{i \leq t} \text{Aut}(M_i),$$

and we deduce (using relation (2) in section 3.1) that

$$(31) \quad [\mathcal{E}_{Q_{\text{Kron}}, \alpha} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] = \prod_{0 \leq i \leq t} [\mathcal{E}_{Q_{\text{Kron}}, (\alpha_i)} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1].$$

If α_i is equal to $a \cdot (n, n \pm 1)$ then

$$(32) \quad [\mathcal{E}_{Q_{\text{Kron}}, (\alpha_i)} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] = [\text{Mat}_{a \times a}(\mathbb{C}) / \text{Gl}_{\mathbb{C}}(a) \xrightarrow{\text{tr}(nX^{d+1} - (n \pm 1)X^{d+1})} \mathbb{A}_{\mathbb{C}}^1]$$

$$(33) \quad = [\text{Mat}_{a \times a}(\mathbb{C}) / \text{Gl}_{\mathbb{C}}(a) \xrightarrow{\text{tr}(X^{d+1})} \mathbb{A}_{\mathbb{C}}^1].$$

Similarly, if $\alpha_i = a \cdot (1, 1)$, then the function $\text{tr}(W_d)$ is zero, restricted to $\mathcal{E}_{Q_{\text{Kron}}, (\alpha_i)}$. This stack is just the stack of length a coherent zero-dimensional sheaves on \mathbb{P}^1 with an endomorphism. It follows that

$$\sum_{a \geq 0} [\mathcal{E}_{Q_{\text{Kron}}, ((a, a))} \rightarrow \mathbb{A}_{\mathbb{C}}^1] \hat{e}_{(a, a)} = \left(\sum_{a \geq 0} [\mathcal{E}_{Q_{\text{Kron}}, ((a, a))}^{\mathbb{C}} \rightarrow \mathbb{A}_{\mathbb{C}}^1] \hat{e}_{(a, a)} \right)^{\mathbb{P}^1},$$

where $\mathcal{E}_{Q_{\text{Kron}}, ((a, a))}^{\mathbb{C}}$ is the stack of pairs (M, ξ) , where M is a coherent $\mathcal{O}_{\mathbb{P}^1}$ -module supported at zero, and ξ is an endomorphism of M . This is just the stack of pairs of commuting matrices N_1 and N_2 such

that N_1 is nilpotent, which in turn is the stack of coherent sheaves on \mathbb{C}^2 supported on zero dimensional subschemes of one of the coordinate lines. As in section 5.6 of [30] one deduces that

$$(34) \quad \sum_{a \geq 0} [\mathcal{E}_{Q_{\text{Kron}}, ((a,a))} \rightarrow \mathbb{A}_{\mathbb{C}}^1] \hat{e}_{(a,a)} = \text{Sym} \left(\sum_{n \geq 1} \frac{\mathbb{L}^{3/2} + \mathbb{L}^{1/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \hat{e}_{(n,n)} \right).$$

Finally, putting all this together, we have

$$\begin{aligned} \Phi_{\text{BBS}, Q_{-2}, W_d}([\mathcal{X}_{Q_{-2}, W_d}]) &= \sum_{\mathbf{n} \in \mathbb{N}^{V(Q_{-2})}} \int_{\mathcal{Y}_{Q_{-2}, \mathbf{n}}} \phi_{\text{tr}(W_d)} \hat{e}_{\mathbf{n}} \\ &= \sum_{\mathbf{n} \in \mathbb{N}^{V(Q_{-2})}} \mathbb{L}^{2\mathbf{n}(0) \cdot \mathbf{n}(1)} [Y_{Q_d, \mathbf{n}} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] \cdot [\text{Gl}_{\mathbb{C}}(\mathbf{n}(0)) \times \text{Gl}_{\mathbb{C}}(\mathbf{n}(1))]^{-1} \hat{e}_{\mathbf{n}} \\ &= \sum_{\mathbf{n} \in \mathbb{N}^{V(Q_{-2})}} [\mathcal{E}_{Q_{\text{Kron}}, \mathbf{n}} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] \hat{e}_{\mathbf{n}} \\ &= \sum_{\alpha \in \mathbb{N}_{\gamma}} [\mathcal{E}_{Q_{\text{Kron}}, \alpha} \xrightarrow{\text{tr}(W_d)} \mathbb{A}_{\mathbb{C}}^1] \hat{e}_{\mathbf{n}} \\ &= \left(\prod_{\mathbf{n}=(n, n \pm 1)} \sum_{a \geq 0} [\text{Mat}_{a \times a}(\mathbb{C}) / \text{Gl}_{\mathbb{C}}(a) \xrightarrow{\text{tr}(X^{d+1})} \mathbb{A}_{\mathbb{C}}^1] \hat{e}_{a\mathbf{n}} \right) \cdot \text{Sym} \left(\sum_{n \geq 1} \frac{\mathbb{L}^{3/2} + \mathbb{L}^{1/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \hat{e}_{(n,n)} \right) \\ &= \text{Sym} \left(\sum_{\mathbf{n}=(n, n \pm 1) \in \mathbb{N}^2} [\mathbb{A}_{\mathbb{C}}^1 \xrightarrow{z \mapsto z^{d+1}} \mathbb{A}_{\mathbb{C}}^1] \cdot \mathbb{L}^{-1/2} (\mathbb{L}^{1/2} - \mathbb{L}^{-1/2})^{-1} \hat{e}_{\mathbf{n}} \right) \\ &\quad \cdot \text{Sym} \left(\sum_{n \geq 1} \frac{\mathbb{L}^{3/2} + \mathbb{L}^{1/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \hat{e}_{(n,n)} \right), \end{aligned}$$

where for the final equality we have again used the calculation of the motivic Donaldson–Thomas invariants for the one loop quiver with potential from [7], and we are done. \square

Remark 5.7. It's possible to give the category of (not necessarily nilpotent) A_{Q_{-2}, W_d} -modules the structure of a cyclic A_{∞} category, and prove the above result in this framework, for arbitrary d .

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